Correct formulation of thermal QCD in the coherent-state representation and rigorous relativistic equation for quark-antiquark bound states at finite temperature

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I. PATH INTEGRAL FORMULATION OF QUANTUM STATISTICS IN THE COHERENT-STATE REPRESENTATION

First, we start from the partition function for a grand canonical ensemble which usually is written in the form

$$Z = Tre^{-\beta\widehat{K}} \tag{1}$$

where $\beta = \frac{1}{kT}$ with k and T being the Boltzmann constant and the temperature and

$$\widehat{K} = \widehat{H} - \mu\widehat{N} \tag{2}$$

here μ is the chemical potential, $\widehat{H}$ and $\widehat{N}$ are the Hamiltonian and particle-number operators respectively. In the coherent-state representation, the trace in Eq. (1) will be represented by an integral over the coherent states. To determine the concrete form of the integral, for simplicity, let us start from an one-dimensional system. Its partition function given in the particle-number representation is

$$Z = \sum_{n=0}^{\infty} \langle n| e^{-\beta\widehat{K}} |n\rangle\,. \tag{3}$$

Then, we use the completeness relation of the coherent states

$$\int D(a^*a) \mid a >< a^* \mid = 1 \tag{4}$$

where $\mid a >$ denotes a normalized coherent state, i.e., the eigenstate of the annihilation operator $\hat{a}$ with a complex eigenvalue a

$$\widehat{a}\,|a\rangle = a\,|a\rangle \tag{5}$$

whose Hermitian conjugate is

$$\langle a^*|\,\widehat{a}^{+} = a^*\,\langle a^*| \tag{6}$$

and $D(a^*a)$ symbolizes the integration measure defined by

$$D(a^*a) = \{ \begin{array}{ll} \frac{1}{\pi}da^*da, & \text{for bosons;} \\ da^*da, & \text{for fermions.} \end{array} \tag{7}$$

In the above, we have used the eigenvalues a and a^* to designate the eigenstates $|a\rangle$ and $\langle a^*|$, respectively. It is emphasized that since we use the normalized eigenfunction of the coherent state whose expression in its own representation will be shown in Eq. (15), the completeness relation in Eq. (4) has the ordinary form as we are familiar with in quantum mechanics. Inserting Eq. (4) into Eq. (3), we have

$$Z = \sum_{n=0}^{\infty} \int D(a^*a)D(a'^*a')\,\langle n \mid a'\rangle\,\langle a'^*|\,e^{-\beta\widehat{K}}\,|a\rangle\,\langle a^* \mid n\rangle \tag{8}$$

where

$$\begin{array}{l}\langle a^* \mid n\rangle = \frac{1}{\sqrt{n!}}(a^*)^n e^{-a^* a}, \\ \langle n \mid a'\rangle = \frac{1}{\sqrt{n!}}(a')^n e^{-a'^* a'}\end{array} \tag{9}$$

are the energy eigenfunctions given in the coherent-state representation (Note: for fermions, $n = 0, 1$). The both eigenfunctions commute with the matrix element $\left\langle a'^* \mid e^{-\beta \widehat{K}} \mid a \right\rangle$ because the operator $\widehat{K}(\widehat{a}^+, \widehat{a})$ generally is a polynomial of the operator $\widehat{a}^+\widehat{a}$ for fermion systems. In view of the expressions in Eq. (9) and the commutation relation

$$a^* a' = \pm a' a^* \tag{10}$$

where the signs "+" and "−" are attributed to bosons and fermions respectively, it is easy to see

$$\langle n \mid a'\rangle \langle a^* \mid n\rangle = \langle \pm a^* \mid n\rangle \langle n \mid a'\rangle . \tag{11}$$

Substituting Eq. (11) in Eq. (8) and applying the completeness relations for the particle-number states and coherent ones, one may find

$$Z = \int D(a^* a) \langle \pm a^* | \, e^{-\beta \widehat{K}} \, | a\rangle \tag{12}$$

where the plus and minus signs in front of a^* belong to bosons and fermions respectively.

To evaluate the matrix element in Eq. (12), we may, as usual, divide the "time" interval $[0, \beta]$ into n equal and infinitesimal parts, $\beta = n\varepsilon$. and then insert a completeness relation shown in Eq. (4) at each dividing point. In this way, Eq. (12) may be represented as

$$\begin{array}{c} Z = \int D(a^* a) \prod\limits_{i=1}^{n-1} D(a_i^* a_i) \langle \pm a^* | \, e^{-\varepsilon \widehat{K}} \, | a_{n-1}\rangle \left\langle a_{n-1}^* \right| e^{-\varepsilon \widehat{K}} \, | a_{n-2}\rangle \cdots \\ \times \left\langle a_{i+1}^* \right| e^{-\varepsilon \widehat{K}} \, | a_i\rangle \langle a_i^* | \, e^{-\varepsilon \widehat{K}} \, | a_{i-1}\rangle \cdots \langle a_1^* | \, e^{-\varepsilon \widehat{K}} \, | a\rangle \end{array} \tag{13}$$

Since ε is infinitesimal, we can write

$$e^{-\varepsilon \widehat{K}(\widehat{a}^+, \widehat{a})} \approx 1 - \varepsilon \widehat{K}(\widehat{a}^+, \widehat{a}) \tag{14}$$

where $\widehat{K}(\widehat{a}^+, \widehat{a})$ is assumed to be normally ordered. Noticing this fact, when applying the equations (5) and (6) and the inner product of two coherent states

$$\langle a_i^* \mid a_{i-1}\rangle = e^{-\frac{1}{2} a_i^* a_i - \frac{1}{2} a_{i-1}^* a_{i-1} + a_i^* a_{i-1}} \tag{15}$$

which suits to the both of bosons and fermions, one can get from Eq. (13) that

$$\begin{array}{c} Z = \int D(a^* a) e^{-a^* a} \int \prod\limits_{i=1}^{n-1} D(a_i^* a_i) \exp\{-\varepsilon \sum\limits_{i=1}^{n} K(a_i^*, a_{i-1}) \\ + \sum\limits_{i=1}^{n} a_i^* a_{i-1} - \sum\limits_{i=1}^{n-1} a_i^* a_i\} \end{array} \tag{16}$$

where we have set

$$\pm a^* = a_n^* \ , \ a = a_0. \tag{17}$$

It is noted that the factor $e^{-a^* a}$ in the first integrand comes from the matrix elements $\langle \pm a^* | \, a_{n-1}\rangle$ and $\langle a_1^* | \, a\rangle$ and the last sum in the above exponent is obtained by summing up the common terms $-\frac{1}{2} a_i^* a_i$ and $-\frac{1}{2} a_{i-1}^* a_{i-1}$ appearing in the exponents of the matrix element $\langle a_i^* \mid a_{i-1}\rangle$ and its adjacent ones $\langle a_{i+1}^* \mid a_i\rangle$ and $\langle a_{i-1}^* \mid a_{i-2}\rangle$. As will be seen in Eq. (21), such a summation is essential to give a correct time-dependence of the functional integrand in the partition function. The last two sums in Eq. (16) can be rewritten in the form

$$\begin{array}{c} \sum\limits_{i=1}^{n} a_i^* a_{i-1} - \sum\limits_{i=1}^{n-1} a_i^* a_i \\ = \frac{1}{2} a_n^* a_{n-1} + \frac{1}{2} a_1^* a_0 + \frac{\varepsilon}{2} \sum\limits_{i=1}^{n-1} [(\frac{a_{i+1}^* - a_i^*}{\varepsilon}) a_i - a_i^* (\frac{a_i - a_{i-1}}{\varepsilon})]. \end{array} \tag{18}$$

Upon substituting Eq. (18) in Eq. (16) and taking the limit $\varepsilon \to 0$, we obtain the path-integral expression of the partition functions as follows:

$$Z = \int D(a^*a)e^{-a^*a} \int \mathfrak{D}(a^*a)e^{I(a^*,a)} \tag{19}$$

where

$$\mathfrak{D}(a^*a) = \{ \begin{array}{ll} \prod_\tau \frac{1}{\pi} da^*(\tau)da(\tau), & \text{for bosons;} \\ \prod_\tau da^*(\tau)da(\tau), & \text{for fermions} \end{array} \tag{20}$$

and

$$\begin{array}{l} I(a^*,a) = \frac{1}{2}a^*(\beta)a(\beta) + \frac{1}{2}a^*(0)a(0) - \int_0^\beta d\tau[\frac{1}{2}a^*(\tau)\dot{a}(\tau) \\ \quad -\frac{1}{2}\dot{a}^*(\tau)a(\tau) + K(a^*(\tau),a(\tau))] \\ = a^*(\beta)a(\beta) - \int_0^\beta d\tau[a^*(\tau)\dot{a}(\tau) + K(a^*(\tau),a(\tau)) \end{array} \tag{21}$$

where the last equality is obtained from the first one by a partial integration. In accordance with the definition given in Eq. (17), we see, the path-integral is subject to the following boundary conditions

$$a^*(\beta) = \pm a^*, a(0) = a \tag{22}$$

where the signs "+" and "−" are written respectively for bosons and fermions. Here it is noted that Eq. (22) does not implies $a(\beta) = \pm a$ and $a^*(0) = a^*$. Actually, we have no such boundary conditions.

For the systems with many degrees of freedom, the functional-integral representation of the partition functions may directly be written out from the results given in Eqs. (19) -(22) as long as the eigenvalues a and a^* are understood as column matrices $a = (a_1, a_2, \cdots, a_k, \cdots)$ and $a^* = (a_1^*, a_2^*, \cdots, a_k^*, \cdots)$. Written explicitly, we have

$$Z = \int D(a^*a)e^{-a_k^*a_k} \int \mathfrak{D}(a^*a)e^{I(a^*,a)} \tag{23}$$

where

$$D(a^*a) = \{ \begin{array}{ll} \prod_k \frac{1}{\pi} da_k^* da_k & \text{, for bosons;} \\ \prod_k da_k^* da_k & \text{, for fermions,} \end{array} \tag{24}$$

$$\mathfrak{D}(a^*a) = \{ \begin{array}{ll} \prod_{k\tau} \frac{1}{\pi} da_k^*(\tau)da_k(\tau) & \text{, for bosons;} \\ \prod_{k\tau} da_k^*(\tau)da_k(\tau) & \text{, for fermions} \end{array} \tag{25}$$

and

$$I(a^*,a) = a_k^*(\beta)a_k(\beta) - \int_0^\beta d\tau[a_k^*(\tau)\dot{a}_k(\tau) + K(a_k^*(\tau),a_k(\tau))]. \tag{26}$$

The boundary conditions in Eq. (22) now become

$$a_k^*(\beta) = \pm a_k^* \ , \ a_k(0) = a_k. \tag{27}$$

In Eqs. (23) and (26), the repeated indices imply the summations over k. If the k stands for a continuous index as in the case of quantum field theory, the summations will be replaced by integrations over k.

It should be pointed out that in the previous derivation of the coherent-state representation of the partition functions, the authors did not use the expressions given in Eqs. (16) and (18). Instead, the matrix element in Eq. (15) was directly chosen to be the starting point and recast in the form

$$\langle a_i^* \mid a_{i-1}\rangle = \exp\{-\frac{\varepsilon}{2}[a_i^*(\frac{a_i - a_{i-1}}{\varepsilon}) - (\frac{a_i^* - a_{i-1}^*}{\varepsilon})a_{i-1}]\}. \tag{28}$$

Substituting the above expression into Eq. (13) and taking the limit $\varepsilon \to 0$, it follows

$$Z = \int \mathfrak{D}(a^*a) \exp\{-\int_0^\beta d\tau[\frac{1}{2}a^*(\tau)\dot{a}(\tau) - \frac{1}{2}\dot{a}^*(\tau)a(\tau) + K(a^*(\tau), a(\tau))]\}. \tag{29}$$

Clearly, in the above derivation, the common terms appearing in the exponents of adjacent matrix elements were not combined together. As a result, the time-dependence of the integrand in Eq. (29) could not be given correctly. In comparison with the previous result shown in Eq. (29), the expression written in Eqs. (19)-(21) has two functional integrals. The first integral which represents the trace in Eq. (1) is absent in Eq. (29). The second integral is defined as the same as the integral in Eq. (29); but the integrand are different from each other. In Eq. (19), there occur two additional factors in the integrand : one is e^{-a^*a} which comes from the initial and final states in Eq. (13), another is $e^{\frac{1}{2}[a^*(\beta)a(\beta)+a^*(0)a(0)]}$ in which $a^*(\beta)$ and $a(0)$ are related to the boundary conditions shown in Eq. (22). These additional factors are also absent in Eq. (29). As will be seen soon later, the occurrence of these factors in the functional-integral expression is essential to give correct calculated results.

To demonstrate the correctness of the expression given in Eqs. (23)-(27), let us compute the partition function for the system whose Hamiltonian is of harmonic oscillator-type as we meet in the cases of ideal gases and free fields. In this case,

$$K(a^*a) = \omega_k a_k^* a_k \tag{30}$$

where $\omega_k = \varepsilon_k - \mu$ with ε_k being the particle energy and therefore Eq. (26) becomes

$$I(a^*, a) = a_k^*(\beta)a_k(\beta) - \int_0^\beta d\tau[a_k^*(\tau)\dot{a}_k(\tau) + \omega_k a_k^*(\tau)a_k(\tau)]. \tag{31}$$

By the stationary-phase method which is established based on the property of the Gaussian integral that the integral is equal to the extremum of the integrand which is an exponential function, we may write

$$\int \mathfrak{D}(a^*a)e^{I(a^*,a)} = e^{I_0(a^*,a)} \tag{32}$$

where $I_0(a^*, a)$ is obtained from $I(a^*, a)$ by replacing the variables $a_k^*(\tau)$ and $a_k(\tau)$ in $I(a^*, a)$ with those values which are determined from the stationary condition $\delta I(a^*, a) = 0$. From this condition and the boundary conditions in Eq. (27) which implies $\delta a_k^*(\beta) = 0$ and $\delta a_k(0) = 0$, it is easy to derive the following equations of motion

$$\dot{a}_k(\tau) + \omega_k a_k(\tau) = 0,\ \dot{a}_k^*(\tau) - \omega_k a_k^*(\tau) = 0. \tag{33}$$

Their solutions satisfying the boundary condition are

$$a_k(\tau) = a_k e^{-\omega_k \tau}\ ,\ a_k^*(\tau) = \pm a_k^* e^{\omega_k(\tau-\beta)}. \tag{34}$$

On substituting the above solutions into Eq. (31), we obtain

$$I_0(a^*, a) = \pm a_k^* a_k e^{-\omega_k \beta} \tag{35}$$

With the functional integral given in Eqs. (32) and (35), the partition functions in Eq. (23) become

$$Z_0 = \left\{ \begin{array}{ll} \int D(a^*a)e^{-a_k^*a_k(1-e^{-\beta\omega_k})} & \text{, for bosons;} \\ \int D(a^*a)e^{-a_k^*a_k(1+e^{-\beta\omega_k})} & \text{, for fermions.} \end{array} \right. \tag{36}$$

For the boson case, the above integral can directly be calculated by employing the integration formula:

$$\int D(a^*a)e^{-a^*(\lambda a - b)}f(a) = \frac{1}{\lambda}f(\lambda^{-1}b) \tag{37}$$

The result is well-known, as shown in the following

$$Z_0 = \prod_k \frac{1}{1 - e^{-\beta\omega_k}} \tag{38}$$

For the fermion case, by using the property of Grassmann algebra and the integration formulas :

$$\int da = \int da^* = 0\ ,\ \int da^*a^* = \int daa = 1 \tag{39}$$

it is easy to compute the integral in Eq. (36) and get the familiar result

$$Z_0 = \prod_k (1 + e^{-\beta\omega_k}) \tag{40}$$

It is noted that if the stationary-phase method is applied to the functional integral in Eq. (29), one could not get the results as written in Eqs. (38) and (40), showing the incorrectness of the previous functional-integral representation for the partition functions.

Now let us turn to discuss the general case where the Hamiltonian can be split into a free part and an interaction part. Correspondingly, we can write

$$K(a^*, a) = K_0(a^*, a) + H_I(a^*, a) \tag{41}$$

where $K_0(a^*, a)$ is the same as given in Eq. (30) and $H_I(a^*, a)$ is the interaction Hamiltonian. In this case, to evaluate the partition function, it is convenient to define a generating functional through introducing external sources $j_k^*(\tau)$ and $j_k(\tau)$ such that

$$\begin{array}{c} Z[j^*, j] = \int D(a^*a) e^{-a*_k a_k} \int \mathfrak{D}(a^*a) \exp\{a_k^*(\beta) a_k(\beta) \\ - \int_0^\beta d\tau [a_k^*(\tau)\dot{a}_k(\tau) + K(a^*a) - j_k^*(\tau) a_k(\tau) - a_k^*(\tau) j_k(\tau)]\} \\ = e^{-\int_0^\beta d\tau H_I(\frac{\delta}{\delta j_k^*(\tau)}, \pm\frac{\delta}{\delta j_k(\tau)})} Z_0[j^*, j] \end{array} \tag{42}$$

where the sings "+" and "−" in front of $\frac{\delta}{\delta j_k(\tau)}$ refer to bosons and fermions respectively and $Z_0[j^*, j]$ is defined by

$$Z_0[j^*, j] = \int D(a^*a) e^{-a_k^* a_k} \int \mathfrak{D}(a^*a) e^{I(a^*, a; j^*, j)} \tag{43}$$

in which

$$\begin{array}{c} I(a^*, a; j^*, j) = a_k^*(\beta) a_k(\beta) - \int_0^\beta d\tau [a_k^*(\tau)\dot{a}_k(\tau) \\ +\omega_k a_k^*(\tau) a_k(\tau) - j_k^*(\tau) a_k(\tau) - a_k^*(\tau) j_k(\tau)] \end{array} \tag{44}$$

Obviously, the integral in Eq. (43) is of Gaussian-type. Therefore, it can be calculated by means of the stationary-phase method as will be shown in detail in Sect. 4.

The exact partition functions can be obtained from the generating functional in Eq. (42) by setting the external sources to be zero

$$Z = Z[j^*, j] \mid_{j^*=j=0} . \tag{45}$$

In particular, the generating functional is much useful to compute the finite-temperature Green functions. For simplicity, we take the two-point Green function as an example to show this point. In many-body theory, the Green function usually is defined in the operator formalism by

$$G_{kl}(\tau_1, \tau_2) = \frac{1}{Z} Tr\{e^{-\beta\widehat{K}} T[\widehat{a}_k(\tau_1)\widehat{a}_l^+(\tau_2)]\} = Tr\{e^{\beta(\Omega - \widehat{K})} T[\widehat{a}_k(\tau_1)\widehat{a}_l^+(\tau_2)]\} \tag{46}$$

where $0 < \tau_1, \tau_2 < \beta$, $\Omega = -\frac{1}{\beta}\ln Z$ is the grand canonical potential, T denotes the "time" ordering operator, $\widehat{a}_k(\tau_1)$ and $\widehat{a}_l^+(\tau_2)$ represent the annihilation and creation operators respectively. According to the procedure described in Eqs. (12)-(22). it is clear to see that when taking τ_1 and τ_2 at two dividing points and applying the equations (5) and (6), the Green function may be expressed as a functional integral in the coherent-state representation as follows

$$G_{kl}(\tau_1, \tau_2) = \frac{1}{Z} \int D(a^*a) e^{-a_k^* a_k} \int \mathfrak{D}(a^*a) a_k(\tau_1) a_l^*(\tau_2) e^{I(a^*, a)}. \tag{47}$$

With the aid of the generating functional defined in Eq. (42), the above Green function may be represented as

$$G_{kl}(\tau_1, \tau_2) = \pm \frac{1}{Z} \frac{\delta^2 Z[j^*, j]}{\delta j_k^*(\tau_1) \delta j_l(\tau_2)} \mid_{j^*=j=0} \tag{48}$$

where the sings "+" and "−" belong to bosons and fermions respectively.

II. GENERATING FUNCTIONAL OF GREEN FUNCTIONS FOR THERMAL QCD

To write out explicitly a path-integral expression of thermal QCD in the coherent-state representation, we first need to formulate the QCD in the coherent-state representation, namely, to give exact expressions of the QCD Hamiltonian and action in the coherent-state representation. For this purpose, we only need to work with the classical fields by using some skilful treatments. Let us start from the effective Lagrangian density of QCD which appears in the path-integral of the zero-temperature QCD

$$\mathcal{L} = \bar{\psi}\{i\gamma^{\mu}(\partial_{\mu} - igT^{a}A_{\mu}^{a}) - m\}\psi - \frac{1}{4}F^{a\mu\nu}F_{\mu\nu}^{a} - \frac{1}{2\alpha}(\partial^{\mu}A_{\mu}^{a})^{2} - \partial^{\mu}\bar{C}^{a}D_{\mu}^{ab}C^{b} \tag{49}$$

where $T^a = \lambda^a/2$ is the color matrix, ψ and $\bar{\psi}$ represent the quark fields, A_μ^a are the vector potentials of gluon fields, C^a and $\bar{C}^a$ designate the ghost fields,

$$F_{\mu\nu}^{a} = \partial_{\mu}A_{\nu}^{a} - \partial_{\nu}A_{\mu}^{a} + gf^{abc}A_{\mu}^{b}A_{\nu}^{c} \tag{50}$$

and

$$D_{\mu}^{ab} = \delta^{ab}\partial_{\mu} - gf^{abc}A_{\mu}^{c} \tag{51}$$

For the sake of simplicity, we work in the Feynman gauge ($\alpha = 1$). It is well-known that in this gauge, the results obtained from the above Lagrangian are equivalent to those derived from the following Lagrangian which is given by applying the Lorentz condition $\partial^{\mu}A_{\mu}^{a} = 0$ to the Lagrangian in Eq. (49),

$$\begin{array}{c}\mathcal{L} = \bar{\psi}\{i\gamma^{\mu}(\partial_{\mu} - igT^{a}A_{\mu}^{a}) - m\}\psi - \frac{1}{2}\partial_{\mu}A_{\nu}^{a}\partial^{\mu}A^{a\nu} - gf^{abc}\partial_{\mu}A_{\nu}^{a}A^{b\mu}A^{c\nu} \\ -\frac{1}{4}g^{2}f^{abc}f^{ade}A^{b\mu}A^{c\nu}A_{\mu}^{d}A_{\nu}^{e} - \partial^{\mu}\bar{C}^{a}\partial_{\mu}C^{b} + gf^{abc}\partial^{\mu}\bar{C}^{a}C^{b}A_{\mu}^{c}\end{array} \tag{52}$$

Here it is noted that the application of the Lorentz condition only changes the form of free part of the gluon Lagrangian, remaining the interaction part of the Lagrangian in Eq. (49) formally unchanged. The above Lagrangian is written in the Minkowski metric where the $\gamma-$matrix is defined as $\gamma_0 = \beta$ and $\vec{\gamma} = \beta\vec{\alpha}$]. In the following, it is convenient to represent the Lagrangian in the Euclidean metric with the imaginary time $\tau = it$ where t is the real time.

Since the path-integral in Eq. (42) is established in the first order (or say, Hamiltonian) formalism, to perform the path-integral quantization of thermal QCD in the coherent-state representation, we need to recast the above Lagrangian in the first order form. In doing this, it is necessary to introduce canonical conjugate momentum densities which are defined by

$$\begin{array}{c}\Pi_{\psi} = \frac{\partial\mathcal{L}}{\partial\partial_{t}\psi} = i\overline{\psi}\gamma^{0} = i\psi^{+}, \\ \Pi_{\overline{\psi}} = \frac{\partial\mathcal{L}}{\partial\partial_{t}\overline{\psi}} = 0, \\ \Pi_{\mu}^{a} = \frac{\partial\mathcal{L}}{\partial\partial_{t}A^{a\mu}} = -\partial_{t}A_{\mu}^{a} + gf^{abc}A_{\mu}^{b}A_{0}^{c}, \\ \Pi^{a} = (\frac{\partial\mathcal{L}}{\partial\partial_{t}C^{a}})_{R} = -\partial_{t}\overline{C}^{a}, \\ \overline{\Pi}^{a} = (\frac{\partial\mathcal{L}}{\partial\partial_{t}\overline{C}^{a}})_{L} = -\partial_{t}C^{a} + gf^{abc}C^{b}A_{0}^{c}\end{array} \tag{53}$$

where the subscripts R and L mark the right and left-derivatives with respect to the real time respectively. With the above momentum densities, the Lagrangian in Eq. (52) can be represented as

$$\mathcal{L} = \Pi_{\psi}\ \partial_{t}\dot{\psi} + \Pi^{a\mu}\partial_{t}A_{\mu}^{a} + \Pi^{a}\partial_{t}C^{a} + \partial_{t}\overline{C}^{a}\overline{\Pi} - \mathcal{H} \tag{54}$$

where

$$\mathcal{H} = \mathcal{H}_{0} + \mathcal{H}_{I} \tag{55}$$

is the Hamiltonian density in which

$$\mathcal{H}_{0} = \bar{\psi}(\vec{\gamma}\cdot\nabla + m)\psi + \frac{1}{2}(\Pi_{\mu}^{a})^{2} - \frac{1}{2}A_{\mu}^{a}\nabla^{2}A_{\mu}^{a} - \Pi^{a}\overline{\Pi}^{a} + \bar{C}^{a}\nabla^{2}C^{a} \tag{56}$$

is the free Hamiltonian density and

$$\begin{array}{c}\mathcal{H}_{I} = ig\bar{\psi}T^{a}\gamma_{\mu}A_{\mu}^{a}\psi + gf^{abc}(i\Pi_{\mu}^{a}A_{4}^{c} + \partial_{i}A_{\mu}^{a}A_{i}^{c})A_{\mu}^{b} - \frac{1}{4}g^{2}f^{abc}f^{ade}A_{\mu}^{b}A_{\mu}^{d} \\ \times(A_{4}^{c}A_{4}^{e} - A_{i}^{c}A_{i}^{e}) + gf^{abc}(i\Pi^{a}A_{4}^{c} - \partial_{i}\bar{C}^{a}A_{i}^{c})C^{b}\end{array} \tag{57}$$

is the interaction Hamiltonian density here the Latin letter i denotes the spatial index. The above Hamiltonian density is written in the Euclidean metric for later convenience. The matrix γ_μ in this metric is defined by $\gamma_4 = \beta$ and $\vec{\gamma} = -i\beta\vec{\alpha}$. It should be noted that the conjugate quantities Π^a and $\overline{\Pi}^a$ for the ghost fields are respectively defined by the right-derivative and the left one as shown in Eq. (53) because only in this way one can get correct results. This unusual definition originates from the peculiar property of the ghost fields which are scalar fields, but subject to the commutation rule of Grassmann algebra.

In order to derive an expression of the thermal QCD in the coherent-state representation, one should employ the Fourier transformations for the canonical variables of the QCD which are listed below. For the quark field,

$$\psi(\vec{x},\tau) = \int \frac{d^3p}{(2\pi)^{3/2}}[u^s(\vec{p})b_s(\vec{p},\tau)e^{i\vec{p}\cdot\vec{x}} + v^s(\vec{p})d_s^*(\vec{p},\tau)e^{-i\vec{p}\cdot\vec{x}}] \tag{58}$$

$$\overline{\psi}(\vec{x},\tau) = \int \frac{d^3p}{(2\pi)^{3/2}}[\overline{u}^s(\vec{p})b_s^*(\vec{p},\tau)e^{-i\vec{p}\cdot\vec{x}} + \overline{v}^s(\vec{p})d_s(\vec{p},\tau)e^{i\vec{p}\cdot\vec{x}}] \tag{59}$$

where $u^s(\vec{p})$ and $v^s(\vec{p})$ are the spinor wave functions satisfying the normalization conditions $u^{s+}(\vec{p})u^s(\vec{p}) = v^{s+}(\vec{p})v^s(\vec{p}) = 1$, $b_s(\vec{p},\tau)$ and $b_s^*(\vec{p},\tau)$ are the eigenvalues of the quark annihilation and creation operators $\widehat{b}_s(\vec{p},\tau)$ and $\widehat{b}_s^+(\vec{p},\tau)$ which are defined in the Heisenberg picture, $d_s(\vec{p},\tau)$ and $d_s^*(\vec{p},\tau)$ are the corresponding ones for antiquarks. For the gluon field,

$$A_\mu^c(\vec{x},\tau) = \int \frac{d^3k}{(2\pi)^{3/2}}\frac{1}{\sqrt{2\omega(\vec{k})}}\varepsilon_\mu^\lambda(\vec{k})[a_\lambda^c(\vec{k},\tau)e^{i\vec{k}\cdot\vec{x}} + a_\lambda^{c*}(\vec{k},\tau)e^{-i\vec{k}\cdot\vec{x}}] \tag{60}$$

where $\varepsilon_\mu^\lambda(\overrightarrow{k})$ is the polarization vector and

$$\Pi_\mu^c(\vec{x},\tau) = i\int \frac{d^3k}{(2\pi)^{3/2}}\sqrt{\frac{\omega(\vec{k})}{2}}\varepsilon_\mu^\lambda(\vec{k})[a_\lambda^c(\vec{k},\tau)e^{i\vec{k}\cdot\vec{x}} - a_\lambda^{c*}(\vec{k},\tau)e^{-i\vec{k}\cdot\vec{x}}] \tag{61}$$

which follows from the definition in Eq. (53) and is consistent with the Fourier representation of free fields. In the above, $a_\lambda^c(\vec{k},\tau)$ and $a_\lambda^{c*}(\vec{k},\tau)$ are the eigenvalues of the gluon annihilation and creation operators $\widehat{a}_\lambda^c(\vec{k},\tau)$ and $\widehat{a}_\lambda^{c+}(\vec{k},\tau)$. For the ghost field, we have

$$\overline{C}^a(\vec{x},\tau) = \int \frac{d^3q}{(2\pi)^{3/2}}\frac{1}{\sqrt{2\omega(\vec{q})}}[\overline{c}_a(\vec{q},\tau)e^{i\vec{q}\cdot\vec{x}} + c_a^*(\vec{q},\tau)e^{-i\vec{q}\cdot\vec{x}}], \tag{62}$$

$$C^a(\vec{x},\tau) = \int \frac{d^3q}{(2\pi)^{3/2}}\frac{1}{\sqrt{2\omega(\vec{q})}}[c_a(\vec{q},\tau)e^{i\vec{q}\cdot\vec{x}} + \overline{c}_a^*(\vec{q},\tau)e^{-i\vec{q}\cdot\vec{x}}], \tag{63}$$

$$\Pi^a(\vec{x},\tau) = i\int \frac{d^3q}{(2\pi)^{3/2}}\sqrt{\frac{\omega(\vec{q})}{2}}[\overline{c}_a(\vec{q},\tau)e^{i\vec{q}\cdot\vec{x}} - c_a^*(\vec{q},\tau)e^{-i\vec{q}\cdot\vec{x}}], \tag{64}$$

and

$$\overline{\Pi}^a(\vec{x},\tau) = i\int \frac{d^3q}{(2\pi)^{3/2}}\sqrt{\frac{\omega(\vec{q})}{2}}[c_a(\vec{q},\tau)e^{i\vec{q}\cdot\vec{x}} - \overline{c}_a^*(\vec{q},\tau)e^{-i\vec{q}\cdot\vec{x}}]. \tag{65}$$

where $c_a(\vec{q},\tau)$ and $c_a^*(\vec{q},\tau)$ are the eigenvalues of the ghost particle annihilation and creation operators $\widehat{c}_a(\vec{q},\tau)$ and $\widehat{c}_a^+(\vec{q},\tau)$ and $\overline{c}_a(\vec{q},\tau)$ and $\overline{c}_a^*(\vec{q},\tau)$ are the ones for antighost particles.

For simplifying the expressions of the Hamiltonian and action of the thermal QCD, it is convenient to use abbreviation notations. Define

$$b_s^\theta(\vec{p},\tau) = \begin{Bmatrix} b_s(\vec{p},\tau), \ if \ \theta = +, \\ d_s^*(\vec{p},\tau), \ if \ \theta = -, \end{Bmatrix} \tag{66}$$

$$W_s^\theta(\vec{p}) = \begin{Bmatrix} (2\pi)^{-3/2} u^s(\vec{p}), \; if \; \theta = +, \\ (2\pi)^{-3/2} v^s(\vec{p}), \; if \; \theta = - \end{Bmatrix} \tag{67}$$

and furthermore, set $\alpha = (\vec{p}, s, \theta)$ and

$$\sum_\alpha = \sum_{s\theta} \int d^3p, \tag{68}$$

Eqs. (58) and (59) may be represented as

$$\begin{aligned} \psi(\vec{x}, \tau) &= \sum_\alpha W_\alpha b_\alpha(\tau) e^{i\theta\vec{p}\cdot\vec{x}}, \\ \overline{\psi}(\vec{x}, \tau) &= \sum_\alpha \overline{W}_\alpha b_\alpha^*(\tau) e^{-i\theta\vec{p}\cdot\vec{x}}. \end{aligned} \tag{69}$$

Similarly, when we define

$$a_{\lambda\theta}^c(\vec{k}, \tau) = \begin{Bmatrix} a_\lambda^c(\vec{k}, \tau), \; if \; \theta = +, \\ a_\lambda^{c*}(\vec{k}, \tau), \; if \; \theta = -, \end{Bmatrix} \tag{70}$$

$$\begin{aligned} A_{\mu\theta}^{c\lambda}(\vec{k}) &= (2\pi)^{-3/2} (2\omega(\vec{k}))^{-1/2} \epsilon_\mu^\lambda(\vec{k}), \\ \Pi_{\mu\theta}^{c\lambda}(\vec{k}) &= i^\theta (2\pi)^{-3/2} [\omega(\vec{q})/2]^{1/2} \epsilon_\mu^\lambda(\vec{k}) \end{aligned} \tag{71}$$

and furthermore, set $\alpha = (\vec{k}, c, \lambda, \theta)$ and

$$\sum_\alpha = \sum_{c\lambda\theta} \int d^3k, \tag{72}$$

Eqs. (60) and (61) can be written as

$$\begin{aligned} A_\mu^c(\vec{x}, \tau) &= \sum_\alpha A_\mu^\alpha a_\alpha(\tau) e^{i\theta\vec{k}\cdot\vec{x}}, \\ \Pi_\mu^c(\vec{x}, \tau) &= \sum_\alpha \Pi_\mu^\alpha a_\alpha(\tau) e^{i\theta\vec{k}\cdot\vec{x}} \end{aligned} \tag{73}$$

For the ghost fields, if we define

$$c_\alpha^\theta(\vec{q}, \tau) = \begin{Bmatrix} \overline{c}_a(\vec{q}, \tau), \; if \; \theta = +, \\ c_a^*(\vec{q}, \tau), \; if \; \theta = -, \end{Bmatrix} \tag{74}$$

$$\begin{aligned} G_\theta(\vec{q}) &= (2\pi)^{-3/2} [2\omega(\vec{q})]^{-1/2}, \\ \Pi_\theta(\vec{q}) &= i^\theta (2\pi)^{-3/2} [\omega(\vec{q})/2]^{1/2}, \end{aligned} \tag{75}$$

and furthermore set $\alpha = (\vec{q}, a, \theta)$ and

$$\sum_\alpha = \sum_{a\theta} \int d^3q \tag{76}$$

then, Eqs. (62)-(65) will be expressed as

$$\begin{aligned} \overline{C}^a(\vec{x}, \tau) &= \sum_\alpha G_\alpha c_\alpha(\tau) e^{i\theta\vec{q}\cdot\vec{x}} \\ C^a(\vec{x}, \tau) &= \sum_\alpha G_\alpha c_\alpha^*(\tau) e^{-i\theta\vec{q}\cdot\vec{x}} \\ \Pi^a(\vec{x}, \tau) &= \sum_\alpha \Pi_\alpha c_\alpha(\tau) e^{i\theta\vec{q}\cdot\vec{x}} \\ \overline{\Pi}^a(\vec{x}, \tau) &= \sum_\alpha \Pi_\alpha c_\alpha^*(\tau) e^{-i\theta\vec{q}\cdot\vec{x}} \end{aligned} \tag{77}$$

Upon substituting Eqs. (69), (73) and (77) into Eqs. (56) and (57), it is not difficult to get

$$H_0(\tau)=\int d^3x\mathcal{H}_0(x)=\sum_\alpha\theta_\alpha\varepsilon_\alpha b^*_\alpha(\tau)b_\alpha(\tau)$$
$$+\frac{1}{2}\sum_\alpha\omega_\alpha a^*_\alpha(\tau)a_\alpha(\tau)+\sum_\alpha\omega_\alpha c^*_\alpha(\tau)c_\alpha(\tau) \tag{78}$$

and

$$H_I(\tau)=\int d^3x\mathcal{H}_I(x)=\sum_{\alpha\beta\gamma}A(\alpha\beta\gamma)b^*_\alpha(\tau)b_\beta(\tau)a_\gamma(\tau)+\sum_{\alpha\beta\gamma}B(\alpha\beta\gamma)a_\alpha(\tau)a_\beta(\tau)a_\gamma(\tau)$$
$$+\sum_{\alpha\beta\gamma\delta}C(\alpha\beta\gamma\delta)a_\alpha(\tau)a_\beta(\tau)a_\gamma(\tau)a_\delta(\tau)+\sum_{\alpha\beta\gamma}D(\alpha\beta\gamma)c^*_\alpha(\tau)c_\beta(\tau)a_\gamma(\tau) \tag{79}$$

which are the QCD Hamiltonian given in the coherent state representation. In Eq. (78), the first, second and third terms are the free Hamiltonians for quarks, gluons and ghost particles respectively where $\theta_\alpha\equiv\theta$, $\varepsilon_\alpha=(\vec{p}^2+m^2)^{1/2}$ is the quark energy, $\omega_\alpha=\left|\vec{k}\right|$ is the energy for a gluon or a ghost particle. In Eq. (79), the first term is the interaction Hamiltonian between quarks and gluons, the second and third terms are the interaction Hamiltonian among gluons and the fourth term represents the interaction Hamiltonian between ghost particles and gluons. The coefficient functions in Eq. (79) are defined as follows:

$$A(\alpha\beta\gamma)=ig(2\pi)^3\delta^3(\theta_\alpha\vec{p}_\alpha-\theta_\beta\vec{p}_\beta-\theta_\gamma\vec{k}_\gamma)\overline{W}^{\theta_\alpha}_{s_\alpha}(\vec{p}_\alpha)T^a\gamma_\mu W^{\theta_\beta}_{s_\alpha}(\vec{p}_\beta)A^{a\lambda_\gamma}_{\mu\theta_\gamma}(\vec{k}_\gamma), \tag{80}$$

$$B(\alpha\beta\gamma)=ig(2\pi)^3\delta^3(\theta_\alpha\vec{k}_\alpha+\theta_\beta\vec{k}_\beta+\theta_\gamma\vec{k}_\gamma)f^{abc}[\Pi^{a\lambda_\alpha}_{\mu\theta_\alpha}(\vec{k}_\alpha)$$
$$\times A^{c\lambda_\gamma}_{4\theta_\gamma}(\vec{k}_\gamma)+\theta_\alpha k^\alpha_i A^{a\lambda_\alpha}_{\mu\theta\alpha}(\vec{k}_\alpha)A^{c\lambda_\gamma}_{i\theta_\gamma}(\vec{k}_\gamma)]A^{b\lambda_\beta}_{\mu\theta_\beta}(\vec{k}_\beta), \tag{81}$$

$$C(\alpha\beta\gamma\delta)=-\frac{1}{4}g^2(2\pi)^3\delta^3(\theta_\alpha\vec{k}_\alpha+\theta_\beta\vec{k}_\beta+\theta_\rho\vec{k}_\rho+\theta_\sigma\vec{k}_\sigma)f^{abc}f^{ade}$$
$$\times A^{b\lambda_\alpha}_{\mu\theta\alpha}(\vec{k}_\alpha)A^{d\lambda_\beta}_{\mu\theta_\beta}(\vec{k}_\beta)[A^{c\lambda_\rho}_{4\theta_\rho}(\vec{k}_\rho)A^{e\lambda_\sigma}_{4\theta_\sigma}(\vec{k}_\sigma)-A^{c\lambda_\rho}_{i\theta_\rho}(\vec{k}_\rho)A^{e\lambda_\sigma}_{i\theta_\sigma}(\vec{k}_\sigma)] \tag{82}$$

and

$$D(\alpha\beta\gamma)=ig(2\pi)^3\delta^3(\theta_\alpha\vec{q}_\alpha-\theta_\beta\vec{q}_\beta-\theta_\gamma\vec{k}_\gamma)f^{abc}G^a_{\theta_\alpha}(\vec{q}_\alpha)$$
$$\times[\Pi^b_{\theta\beta}(\vec{q}_\beta)A^{c\lambda_\gamma}_{4\theta_\gamma}(\vec{k}_\gamma)-\theta_\alpha k^\alpha_i G^b_{\theta_\beta}(\vec{q}_\beta)A^{c\lambda_\gamma}_{i\theta_\gamma}(\vec{k}_\gamma)]. \tag{83}$$

It is emphasized that the expressions in Eqs. (78) and (79) are just the Hamiltonian of QCD appearing in the path-integral as shown in Eq. (42) where all the creation and annihilation operators in the Hamiltonian (which are written in a normal product) are replaced by their eigenvalues.

To write the path-integral of thermal QCD, we need also an expression of action S given in the coherent state representation. This action can be obtained by using the Lagrangian density shown in Eq. (54). By partial integration and considering the following boundary conditions of the fields:

$$\psi(\vec{x},0)=\psi(\vec{x}),\ \overline{\psi}(\vec{x},0)=\overline{\psi}(\vec{x}),$$
$$\psi(\vec{x},\beta)=-\psi(\vec{x}),\ \overline{\psi}(\vec{x},\beta)=-\overline{\psi}(\vec{x}), \tag{84}$$

$$A^c_\mu(\vec{x},0)=A^c_\mu(\vec{x},\beta)=A^c_\mu(\vec{x}),$$
$$\Pi^c_\mu(\vec{x},0)=\Pi^c_\mu(\vec{x},\beta)=\Pi^c_\mu(\vec{x}) \tag{85}$$

and

$$\overline{C}^a(\vec{x},0)=\overline{C}^a(\vec{x},\beta)=\overline{C}^a(\vec{x}),\ C^a(\vec{x},0)=C^a(\vec{x},\beta)=C^a(\vec{x}),$$
$$\overline{\Pi}^a(\vec{x},0)=\overline{\Pi}^a(\vec{x},\beta)=\overline{\Pi}^a(\vec{x}),\ \Pi^a(\vec{x},0)=\Pi^a(\vec{x},\beta)=\Pi^a(\vec{x}), \tag{86}$$

the action given by the Lagrangian density in Eq. (54) can be represented in the form

$$S=\int_0^\beta d\tau\int d^3x\{\frac{1}{2}[\psi^+(\vec{x},\tau)\dot{\psi}(\vec{x},\tau)-\dot{\psi}^+(\vec{x},\tau)\psi(\vec{x},\tau)]$$
$$+\frac{i}{2}[\Pi^c_\mu(\vec{x},\tau)\dot{A}^c_\mu(\vec{x},\tau)-\dot{\Pi}^c_\mu(\vec{x},\tau)A^c_\mu(\vec{x},\tau)]$$
$$+\frac{i}{2}[\Pi_a(\vec{x},\tau)\dot{C}_a(\vec{x},\tau)-\dot{\Pi}_a(\vec{x},\tau)C_a(\vec{x},\tau)$$
$$+\dot{\overline{C}}_a(\vec{x},\tau)\overline{\Pi}_a(\vec{x},\tau)-\overline{C}_a(\vec{x},\tau)\dot{\overline{\Pi}}_a(\vec{x},\tau)]-\mathcal{H}(\vec{x},\tau)\} \tag{87}$$

where the first relation in Eq. (53) has been used and the symbol " $\cdot$ " in $\dot{\psi}(\vec{x},\tau), \dot{A}^c_\mu(\vec{x},\tau)\cdots\cdots$ now denotes the derivatives of the fields with respect to the imaginary time τ. It is stressed here that only the above expression is appropriate to use for deriving the coherent-state representation of the action by making use of the Fourier expansions written in Eqs. (58)-(65). On inserting Eqs. (58)-(65) into Eq. (87), it is not difficult to get

$$\begin{array}{c} S=-\int_0^\beta d\tau\{\int d^3k\{\frac{1}{2}[b_s^*(\vec{k},\tau)\dot{b}_s(\vec{k},\tau)-\dot{b}_s^*(\vec{k},\tau)b_s(\vec{k},\tau)]+\frac{1}{2}[d_s^*(\vec{k},\tau)\dot{d}_s(\vec{k},\tau) \\ -\dot{d}_s^*(\vec{k},\tau)d_s(\vec{k},\tau)]+\frac{1}{2}[a_\lambda^{c*}(\vec{k},\tau)\dot{a}_\lambda^c(\vec{k},\tau)-\dot{a}_\lambda^{c*}(\vec{k},\tau)a_\lambda^c(\vec{k},\tau)]+\frac{1}{2}[\overline{c}_a^*(\vec{k},\tau)\dot{\overline{c}}_a(\vec{k},\tau) \\ -\dot{\overline{c}}_a^*(\vec{k},\tau)\overline{c}_a(\vec{k},\tau)-c_a^*(\vec{k},\tau)\dot{c}_a(\vec{k},\tau)+\dot{c}_a^*(\vec{k},\tau)c_a(\vec{k},\tau)]\}+H(\tau)\} \\ =-S_E \end{array} \tag{88}$$

where $H(\tau)$ is given by the sum of the Hamiltonians in Eqs. (78) and (79) and S_E is the action defined in the Euclidean metric. It is noted that if one considers a grand canonical ensemble of QCD, the Hamiltonian in Eq. (88) should be replaced by $K(\tau)$ defined in Eq. (2). Employing the abbreviation notation as denoted in Eqs. (66), (70) and (74) and letting q_α stand for $(a_\alpha, b_\alpha, c_\alpha)$, the action may be compactly represented as

$$S_E=\int_0^\beta d\tau\{\sum_\alpha\frac{1}{2}[q_\alpha^*(\tau)\circ\dot{q}_\alpha(\tau)-\dot{q}_\alpha^*(\tau)\circ q_\alpha(\tau)]+H(\tau)\} \tag{89}$$

where we have defined

$$q_\alpha^*\circ q_\alpha=a_{\alpha^-}a_{\alpha^+}+b_\alpha^*b_\alpha+\theta_\alpha c_\alpha^*c_\alpha \tag{90}$$

It is emphasized that the $\theta_\alpha=\pm$ is now contained in the subscript α. Therefore, each α may takes α^+ and/or α^- as the first term in Eq. (90) does.

With the action S_E given in the preceding section, the quantization of the thermal QCD in the coherent-state representation is easily implemented by writing out its generating functional of thermal Green functions. According to the general formula shown in Eq. (42), the QCD generating functional can be formulated as

$$\begin{array}{c} Z[j]=\int D(q^*q)e^{-q^*\cdot q}\int\mathfrak{D}(q^*q)\exp\{\frac{1}{2}[q^*(\beta)\cdot q(\beta) \\ -q^*(0)\cdot q(0)]-S_E+\int_0^\beta d\tau j^*(\tau)\cdot q(\tau)\} \end{array} \tag{91}$$

where we have defined

$$q^*\cdot q=\frac{1}{2}a_\alpha^*a_\alpha+\theta_\alpha b_\alpha^*b_\alpha+c_\alpha^*c_\alpha \tag{92}$$

and

$$j^*\cdot q=\xi_\alpha^*a_\alpha+\theta_\alpha(\eta_\alpha^*b_\alpha+b_\alpha^*\eta_\alpha+\zeta_\alpha^*c_\alpha+c_\alpha^*\zeta_\alpha) \tag{93}$$

here ξ_α, η_α and ζ_α are the sources for gluons, quarks and ghost particles respectively and the repeated index implies summation. It is noted that the product $q^*\cdot q$ defined above is different from the $q_\alpha^*\circ q_\alpha$ defined in Eq. (90) in the terms for quarks and ghost particles and the subscript α in Eqs. (92) and (93) is also defined by containing $\theta_\alpha=\pm$. In what follows, we assign $\alpha^\pm$ to represent the α with $\theta_\alpha=\pm$. According to this notation, the sources in Eq. (93) are specifically defined as follows:

$$\begin{array}{c} \xi_{\alpha^+}=\xi_\alpha,\ \xi_{\alpha^-}=\xi_\alpha^* \\ \eta_{\alpha^+}=\eta_\alpha,\ \eta_{\alpha^-}=\overline{\eta}_\alpha^* \\ \zeta_{\alpha^+}=\zeta_\alpha,\ \zeta_{\alpha^-}=\overline{\zeta}_\alpha^* \end{array} \tag{94}$$

where the subscript α on the right hand side of each equality no longer contains θ_α and the gluon term in Eq. (92) $(1/2)a_\alpha^*a_\alpha$ may be replaced by $a_{\alpha^-}a_{\alpha^+}$. The integration measures $D(q^*q)$ and $\mathfrak{D}(q^*q)$ are defined as in Eqs. (24) and (25).

III. RELATIVISTIC EQUATION FOR $Q\overline{Q}$ BOUND STATES

With the generating functional given in the preceding section, we are ready to derive the relativistic equation for $q\overline{q}$ bound states at finite temperature. It is well-known that a bound state exists in the space-like Minkowski space in which there always is an equal-time Lorentz frame. Since in the equal-time frame, the relativistic equation is reduced to a three-dimensional one without loss of any rigorism, in this section we only pay our attention to the three-dimensional equation which may be derived from the equations of motion satisfied by the following $q\overline{q}$ two-"time" (temperature) four-point Green function

$$\begin{aligned}\mathcal{G}(\alpha\beta;\gamma\delta;\tau_1-\tau_2) &= Tr\{e^{\beta(\Omega-\widehat{K})}T\{N[\widehat{b}_\alpha(\tau_1)\widehat{b}^+_\beta(\tau_1)]N[\widehat{b}_\gamma(\tau_2)\widehat{b}^+_\delta(\tau_2)]\}\} \\ &\equiv \left\langle T\{N[\widehat{b}_\alpha(\tau_1)\widehat{b}^+_\beta(\tau_1)]N[\widehat{b}_\gamma(\tau_2)\widehat{b}^+_\delta(\tau_2)]\}\right\rangle_\beta\end{aligned} \tag{95}$$

where the symbol $\langle\cdots\rangle_\beta$ represents the statistical average and N symbolizes the normal product whose definition can be given from the corresponding definition at zero-temperature by replacing the vacuum average with the statistical average

$$N[\widehat{b}_\alpha(\tau_1)\widehat{b}^+_\beta(\tau_2)] = T[\widehat{b}_\alpha(\tau_1)\widehat{b}^+_\beta(\tau_2)] - S_{\alpha\beta}(\tau_1-\tau_2) \tag{96}$$

where

$$S_{\alpha\beta}(\tau_1-\tau_2) = \left\langle T[\widehat{b}_\alpha(\tau_1)\widehat{b}^+_\beta(\tau_2)]\right\rangle_\beta \tag{97}$$

is the quark or antiquark thermal propagator. The normal product in Eq. (95) plays a role of excluding the contraction between the quark and the antiquark operators from the Green function when the quark and antiquark are of the same flavor. Physically, this avoids the $q\overline{q}$ annihilation that would break stability of a bound state. Substituting Eq. (96) in Eq. (95), we have

$$\mathcal{G}(\alpha\beta;\gamma\delta;\tau_1-\tau_2) = G(\alpha\beta;\gamma\delta;\tau_1-\tau_2) - S_{\alpha\beta}S_{\gamma\delta} \tag{98}$$

where

$$G(\alpha\beta;\gamma\delta;\tau_1-\tau_2) = \left\langle T\{\widehat{b}_\alpha(\tau_1)\widehat{b}^+_\beta(\tau_1)\widehat{b}_\gamma(\tau_2)\widehat{b}^+_\delta(\tau_2)\}\right\rangle_\beta \tag{99}$$

is the ordinary Green function and, $S_{\alpha\beta}$ and $S_{\gamma\delta}$ are the equal-time quark (antiquark) propagators. Obviously, in order to derive the equation of motion satisfied by the Green function $\mathcal{G}(\alpha\beta;\gamma\delta;\tau_1-\tau_2)$, we need first to derive the equation of motion for the Green function $G(\alpha\beta;\gamma\delta;\tau_1-\tau_2)$.

Let us start with the generating functional in Eq. (91). By partial integration of the second term on the right hand side of Eq. (89), the generating functional may be written in the form

$$\begin{aligned}Z[j] = \int D(q^*q)e^{-q^*\cdot q}\int \mathfrak{D}(q^*q)\exp\{q^*(\beta)\cdot q(\beta) \\ -S_E + \textstyle\int_0^\beta d\tau j^*(\tau)\cdot q(\tau)\}\end{aligned} \tag{100}$$

where

$$S_E = \int_0^\beta d\tau\{\sum_\alpha q^*_\alpha(\tau)\circ\dot{q}_\alpha(\tau) + H(\tau)\} \tag{101}$$

here $H(\tau)$ was given in Eqs. (78) and (79). First, we derive an equation of motion describing the variation of the $q\overline{q}$ four-point Green function with the "time" variable τ_1. For this purpose, let us differentiate the generating functional in Eq. (100) with respect to $b^*_\alpha(\tau_1)$. Considering that the generating functional is independent of $b^*_\alpha(\tau_1)$ and noticing the expressions given in Eqs. (101), (78), (79) and (93), one may obtain

$$\begin{aligned}\tfrac{\delta Z[j]}{\delta b^*_\alpha(\tau_1)} = \int D(q^*q)e^{-q^*\cdot q}\int \mathfrak{D}(q^*q)[-\dot{b}_\alpha(\tau_1) - \varepsilon_\alpha\theta_\alpha b_\alpha(\tau_1) \\ -\sum_{\rho\lambda}A_{\alpha\rho\lambda}b_\rho(\tau_1)a_\lambda(\tau_1) + \theta_\alpha\eta_\alpha(\tau_1)]\exp\{q^*(\beta)\cdot q(\beta) - S_E \\ -\textstyle\int_0^\beta d\tau j^*(\tau)\cdot q(\tau)\} = 0.\end{aligned} \tag{102}$$

When the $b_\alpha(\tau_1)$ and $a_\lambda(\tau_1)$ in the above are replaced by the functional derivatives $\theta_\alpha\delta/\delta\eta^*_\alpha(\tau_1)$ and $\delta/\delta j^*_\lambda(\tau_1)$ respectively and multiplying the both sides of Eq. (102) with θ_α, the above equation can be written as

$$\{\frac{d}{d\tau_1}\frac{\delta}{\delta\eta^*_\alpha(\tau_1)} + \theta_\alpha\varepsilon_\alpha\frac{\delta}{\delta\eta^*_\alpha(\tau_1)} + \sum_{\rho\lambda}\theta_\alpha\theta_\rho A(\alpha\rho\lambda)\frac{\delta^2}{\delta\eta^*_\rho(\tau_1)\delta j^*_\lambda(\tau_1)} - \eta_\alpha(\tau_1)\}Z[j] = 0. \tag{103}$$

Then, we differentiate the above equation with respect to the sources $\eta_\beta((\tau_1)$, giving

$$\begin{aligned}\{(\tfrac{d}{d\tau_1}\tfrac{\delta}{\delta\eta^*_\alpha(\tau_1)})\tfrac{\delta}{\delta\eta_\beta(\tau_1)}+\theta_\alpha\varepsilon_\alpha\tfrac{\delta^2}{\delta\eta^*_\alpha(\tau_1)\delta\eta_\beta(\tau_1)}+\sum_{\rho\lambda}\theta_\alpha\theta_\rho A(\alpha\rho\lambda)\tfrac{\delta^3}{\delta\eta^*_\rho(\tau_1)\delta\eta_\beta(\tau_1)\delta j^*_\lambda(\tau_1)}\\+\delta_{\alpha\beta}-\eta_\alpha(\tau_1)\tfrac{\delta}{\delta\eta_\beta(\tau_1)}\}Z[j]=0.\end{aligned}\tag{104}$$

Furthermore, successive differentiations of Eq. (104) with respect sources $\eta^*_\gamma(\tau_2)$ and $\eta_\delta(\tau_2)$ yield

$$\begin{aligned}&\{(\tfrac{d}{d\tau_1}\tfrac{\delta}{\delta\eta^*_\alpha(\tau_1)})\tfrac{\delta^3}{\delta\eta_\beta(\tau_1)\delta\eta^*_\gamma(\tau_2)\delta\eta_\delta(\tau_2)}+\theta_\alpha\varepsilon_\alpha\tfrac{\delta^4}{\delta\eta^*_\alpha(\tau_1)\delta\eta_\beta(\tau_1)\delta\eta^*_\gamma(\tau_2)\delta\eta_\delta(\tau_2)}\\&+\sum_{\lambda\sigma}\theta_\alpha\theta_\rho A(\alpha\rho\lambda)\tfrac{\delta^5}{\delta\eta^*_\rho(\tau_1)\delta\eta_\beta(\tau_1)\delta\eta^*_\gamma(\tau_2)\delta\eta_\delta(\tau_2)\delta j^*_\lambda(\tau_1)}+\delta_{\alpha\beta}\tfrac{\delta^2}{\delta\eta^*_\gamma(\tau_2)\delta\eta_\delta(\tau_2)}\\&-\delta_{\alpha\delta}\delta(\tau_1-\tau_2)\tfrac{\delta^2}{\delta\eta^*_\gamma(\tau_2)\delta\eta_\beta(\tau_1)}-\eta_\alpha(\tau_1)\tfrac{\delta^3}{\delta\eta_\beta(\tau_1)\delta\eta^*_\gamma(\tau_2)\delta\eta_\delta(\tau_2)}\}Z[j]=0.\end{aligned}\tag{105}$$

Similarly, when differentiating Eq. (100) with respect $b_\beta(\tau_1)$, one may obtain

$$\{\frac{d}{d\tau_1}\frac{\delta}{\delta\eta_\beta(\tau_1)}-\theta_\beta\varepsilon_\beta\frac{\delta}{\delta\eta_\beta(\tau_1)}-\sum_{\sigma\lambda}\theta_\sigma\theta_\beta A(\sigma\beta\lambda)\frac{\delta^2}{\delta\eta_\sigma(\tau_1)\delta j^*_\lambda(\tau_1)}-\eta^*_\beta(\tau_1)\}Z[j]=0,\tag{106}$$

Subsequently, On differentiating the above equation with respect to $\eta^*_\alpha(\tau_1)$, we get

$$\begin{aligned}\{\tfrac{\delta}{\delta\eta^*_\alpha(\tau_1)}(\tfrac{d}{d\tau_1}\tfrac{\delta}{\delta\eta_\beta(\tau_1)})-\theta_\beta\varepsilon_\beta\tfrac{\delta^2}{\delta\eta^*_\alpha(\tau_1)\delta\eta_\beta(\tau_1)}-\sum_{\sigma\lambda}\theta_\beta\theta_\sigma A(\sigma\beta\lambda)\tfrac{\delta^3}{\delta\eta^*_\alpha(\tau_1)\delta\eta_\sigma(\tau_1)\delta j^*_\lambda(\tau_1)}\\-\delta_{\alpha\beta}+\eta^*_\beta(\tau_1)\tfrac{\delta}{\delta\eta^*_\alpha(\tau_1)}\}Z[j]=0.\end{aligned}\tag{107}$$

Finally, successive differentiations of the above equation with respect to the sources $\eta^*_\gamma(\tau_2)$ and $\eta_\delta(\tau_2)$ give rise to

$$\begin{aligned}&\{\tfrac{\delta}{\delta\eta^*_\alpha(\tau_1)}(\tfrac{d}{d\tau_1}\tfrac{\delta}{\delta\eta_\beta(\tau_1)})\tfrac{\delta^2}{\delta\eta^*_\gamma(\tau_2)\delta\eta_\delta(\tau_2)}-\theta_\beta\varepsilon_\beta\tfrac{\delta^4}{\delta\eta^*_\alpha(\tau_1)\delta\eta_\beta(\tau_1)\delta\eta^*_\gamma(\tau_2)\delta\eta_\delta(\tau_2)}\\&-\sum_{\lambda\sigma}\theta_\beta\theta_\sigma A(\sigma\beta\lambda)\tfrac{\delta^5}{\delta\eta^*_\alpha(\tau_1)\delta\eta_\sigma(\tau_1)\delta\eta^*_\gamma(\tau_2)\delta\eta_\delta(\tau_2)\delta j^*_\lambda(\tau_1)}-\delta_{\alpha\beta}\tfrac{\delta^2}{\delta\eta^*_\gamma(\tau_2)\delta\eta_\delta(\tau_2)}\\&+\delta_{\beta\gamma}\delta(\tau_1-\tau_2)\tfrac{\delta^2}{\delta\eta^*_\alpha(\tau_1)\delta\eta_\delta(\tau_2)}+\eta^*_\beta(\tau_1)\tfrac{\delta^3}{\delta\eta^*_\alpha(\tau_1)\delta\eta^*_\gamma(\tau_2)\delta\eta_\delta(\tau_2)}\}Z[j]=0.\end{aligned}\tag{108}$$

Adding Eq. (104) to Eq. (107), then multiplying the both sides of the equation thus obtained with $-\theta_\alpha\theta_\beta$ and finally setting the external sources $\eta^*_\alpha=\eta_\beta=0$, but remaining the gluon source $j_\lambda\neq 0$, we get

$$(\frac{d}{d\tau_1}+\theta_\alpha\varepsilon_\alpha-\theta_\beta\varepsilon_\beta)S^{j_\lambda}_{\alpha\beta}+\sum_{\rho\sigma\lambda}[A(\alpha\rho\lambda)\delta_{\beta\sigma}-A(\sigma\beta\lambda)\delta_{\alpha\rho}]\frac{\delta}{\delta j^*_\lambda(\tau_1)}S^{j_\lambda}_{\rho\sigma}=0\tag{109}$$

where

$$S^{j_\lambda}_{\alpha\beta}=-\frac{1}{Z}\theta_\alpha\theta_\beta\frac{\delta^2 Z[j]}{\delta\eta^*_\alpha(\tau_1)\delta\eta_\beta(\tau_1)}\mid_{\eta^*_\alpha=\eta_\beta=0}\tag{110}$$

is the quark (antiquark) equal-time propagator in the presence of source j_λ. If we define

$$H(\alpha\beta;\rho\sigma;\tau_1)^{j_\lambda}=(\frac{d}{d\tau_1}+\theta_\alpha\varepsilon_\alpha-\theta_\beta\varepsilon_\beta)\delta_{\alpha\rho}\delta_{\beta\sigma}+\sum_\lambda f(\alpha\beta;\rho\sigma\lambda)\frac{\delta}{\delta j^*_\lambda(\tau_1)}\tag{111}$$

where

$$f(\alpha\beta;\rho\sigma\lambda)=A(\alpha\rho\lambda)\delta_{\beta\sigma}-A(\sigma\beta\lambda)\delta_{\alpha\rho},\tag{112}$$

Eq. (109) can simply be represented as

$$\sum_{\rho\sigma}H(\alpha\beta;\rho\sigma;\tau_1)^{j_\lambda}S^{j_\lambda}_{\rho\sigma}=0.\tag{113}$$

When summing up the both equations in Eqs. (105) and (108), then multiplying the equation thus obtained with $\theta_\alpha\theta_\beta\theta_\gamma\theta_\delta$ and finally setting all the sources but j_λ to be zero, one may get

$$(\frac{d}{d\tau_1}+\theta_\alpha\varepsilon_\alpha-\theta_\beta\varepsilon_\beta)G(\alpha\beta;\gamma\delta;\tau_1-\tau_2)^{j_\lambda}+\sum_{\rho\sigma\lambda}f(\alpha\beta;\rho\sigma\lambda)\frac{\delta}{\delta j_\lambda^*(\tau_1)}G(\rho\sigma;\gamma\delta;\tau_1-\tau_2)^{j_\lambda}$$
$$=\delta(\tau_1-\tau_2)[\delta_{\beta\gamma}S_{\alpha\delta}(\tau_1-\tau_2)^{j_\lambda}-\delta_{\alpha\delta}S_{\gamma\beta}(\tau_2-\tau_1)^{j_\lambda}] \tag{114}$$

where

$$G(\alpha\beta;\gamma\delta;\tau_1-\tau_2)^{j_\lambda}=\frac{1}{Z}\theta_\alpha\theta_\beta\theta_\gamma\theta_\delta\frac{\delta^4 Z[j]}{\delta\eta_\alpha^*(\tau_1)\delta\eta_\beta(\tau_1)\delta\eta_\gamma^*(\tau_2)\delta\eta_\delta(\tau_2)}\mid_{\eta^*=\eta=0} \tag{115}$$

and

$$S_{\alpha\beta}(\tau_1-\tau_2)^{j_\lambda}=-\frac{1}{Z}\theta_\alpha\theta_\beta\frac{\delta^2 Z[j]}{\delta\eta_\alpha^*(\tau_1)\delta\eta_\beta(\tau_2)}\mid_{\eta^*=\eta=0} \tag{116}$$

are respectively the $q\overline{q}$ two-"time" four-point thermal Green function and the quark or antiquark thermal propagator in presence of source j_λ. When the source j_λ is turned off, Eqs. (115) and (116) will respectively go over to the Green function in Eq. (99) and the propagator in Eq. (97). It is noted that due to the restriction of the delta function, the propagators in Eq. (114) are actually "time"-independent. With the definition in Eq. (111), Eq. (114) may be represented as

$$\sum_{\rho\sigma}H(\alpha\beta;\rho\sigma;\tau_1)^{j_\lambda}G(\rho\sigma;\gamma\delta;\tau_1-\tau_2)^{j_\lambda}=-\delta(\tau_1-\tau_2)S(\alpha\beta;\gamma\delta)^{j_\lambda} \tag{117}$$

where

$$S(\alpha\beta;\gamma\delta)^{j_\lambda}=\delta_{\alpha\delta}S_{\gamma\beta}^{j_\lambda}-\delta_{\beta\gamma}S_{\alpha\delta}^{j_\lambda}. \tag{118}$$

Acting on the both sides of Eq. (155) with the operator $H(\alpha\beta;\rho\sigma;\tau_1)^{j_\lambda}$ and using the equations in Eqs. (113) and (117), we find

$$\sum_{\rho\sigma}H(\alpha\beta;\rho\sigma;\tau_1)^{j_\lambda}\mathcal{G}(\rho\sigma;\gamma\delta;\tau_1-\tau_2)^{j_\lambda}=\sum_{\rho\sigma}H(\alpha\beta;\rho\sigma;\tau_1)^{j_\lambda}G(\rho\sigma;\gamma\delta;\tau_1-\tau_2)^{j_\lambda}$$
$$=-\delta(\tau_1-\tau_2)S(\alpha\beta;\gamma\delta)^{j_\lambda} \tag{119}$$

This indicates that the equation of motion satisfied by the Green function $\mathcal{G}(\alpha\beta;\gamma\delta;\tau_1-\tau_2)$ formally is the same as the one shown in Eq. (114). Therefore, in the case that the source j_λ vanishes, we can write

$$(\frac{d}{d\tau_1}+\theta_\alpha\varepsilon_\alpha-\theta_\beta\varepsilon_\beta)\mathcal{G}(\alpha\beta;\gamma\delta;\tau_1-\tau_2)$$
$$=-\delta(\tau_1-\tau_2)S(\alpha\beta;\gamma\delta)-\sum_{\rho\sigma\lambda}f(\alpha\beta;\rho\sigma\lambda)\mathcal{G}(\rho\sigma\lambda;\gamma\delta;\tau_1-\tau_2) \tag{120}$$

where

$$\mathcal{G}(\rho\sigma\lambda;\gamma\delta;\tau_1-\tau_2)=\frac{\delta}{\delta j_\lambda^*(\tau_1)}\mathcal{G}(\rho\sigma;\gamma\delta;\tau_1-\tau_2)^{j_\lambda}\mid_{j_\lambda=0}$$
$$=\left\langle T\{N[\widehat{b}_\rho(\tau_1)\widehat{b}_\sigma^+(\tau_1)\widehat{a}_\lambda(\tau_1)]N[\widehat{b}_\gamma(\tau_2)\widehat{b}_\delta^+(\tau_2)]\}\right\rangle_\beta \tag{121}$$

and

$$S(\alpha\beta;\gamma\delta)=\delta_{\alpha\delta}S_{\gamma\beta}-\delta_{\beta\gamma}S_{\alpha\delta}=-\langle[\widehat{b}_\alpha\widehat{b}_\beta^+,\widehat{b}_\gamma\widehat{b}_\delta^+]_-\rangle_\beta. \tag{122}$$

It is noted here that similar to the definition in Eq. (96), the normal product $N[\widehat{b}_\rho(\tau_1)\widehat{b}_\sigma^+(\tau_1)\widehat{a}_\lambda(\tau_1)]$ in Eq. (121) is defined as

$$N[\widehat{b}_\rho(\tau_1)\widehat{b}_\sigma^+(\tau_1)\widehat{a}_\lambda(\tau_1)]=T[\widehat{b}_\rho(\tau_1)\widehat{b}_\sigma^+(\tau_1)\widehat{a}_\lambda(\tau_1)]-\Lambda(\rho\sigma\lambda) \tag{123}$$

where

$$\Lambda(\rho\sigma\lambda)=\left\langle T[\widehat{b}_\rho(\tau_1)\widehat{b}_\sigma^+(\tau_1)\widehat{a}_\lambda(\tau_1)]\right\rangle_\beta. \tag{124}$$

Substituting Eqs. (96) and (1123) into Eq. (121), we have

$$\mathcal{G}(\rho\sigma\lambda;\gamma\delta;\tau_1-\tau_2)=G(\rho\sigma\lambda;\gamma\delta;\tau_1-\tau_2)-\Lambda(\rho\sigma\lambda)S_{\gamma\delta} \tag{125}$$

where

$$G(\rho\sigma\lambda;\gamma\delta;\tau_1-\tau_2)=\left\langle T\{\widehat{b}_\rho(\tau_1)\widehat{b}_\sigma^+(\tau_1)\widehat{a}_\lambda(\tau_1)\widehat{b}_\gamma(\tau_2)\widehat{b}_\delta^+(\tau_2)\}\right\rangle_\beta \tag{126}$$

is the ordinary five-point thermal Green function including a gluon operator in it.

By the well-known argument, it is easy to prove that the Green functions $\mathcal{G}(\alpha\beta;\gamma\delta;\tau_1-\tau_2)$ and $\mathcal{G}(\rho\sigma\lambda;\gamma\delta;\tau_1-\tau_2)$ are periodic. Therefore, we have the following Fourier expansions:

$$\begin{aligned}\mathcal{G}(\alpha\beta;\gamma\delta;\tau)&=\tfrac{1}{\beta}\sum_n\mathcal{G}(\alpha\beta;\gamma\delta;\omega_n)e^{-i\omega_n\tau},\\ \mathcal{G}(\rho\sigma\lambda;\gamma\delta;\tau)&=\tfrac{1}{\beta}\sum_n\mathcal{G}(\rho\sigma\lambda;\gamma\delta;\omega_n)e^{-i\omega_n\tau}\end{aligned} \tag{127}$$

where $\tau=\tau_1-\tau_2$ and $\omega_n=\frac{2\pi n}{\beta}$. Upon inserting Eq. (127) into Eq. (120) and performing the integration $\frac{1}{2}\int_{-\beta}^{\beta}d\tau e^{i\omega_n\tau}$, we arrive at

$$\begin{aligned}&(i\omega_n-\theta_\alpha\varepsilon_\alpha+\theta_\beta\varepsilon_\beta)\mathcal{G}(\alpha\beta;\gamma\delta;\omega_n)\\ &=-S(\alpha\beta;\gamma\delta)+\sum_{\rho\sigma\lambda}f(\alpha\beta;\rho\sigma\lambda)\mathcal{G}(\rho\sigma\lambda;\gamma\delta;\omega_n).\end{aligned} \tag{128}$$

It is well-known that the Green function $\mathcal{G}(\rho\sigma\lambda;\gamma\delta;\omega_n)$ is B-S (two-particle) reducible. Therefore, we can write

$$\sum_{\lambda\tau\rho}f(\alpha\beta;\rho\sigma\lambda)\mathcal{G}(\rho\sigma\lambda;\gamma\delta;\omega_n)=\sum_{\mu\nu}K(\alpha\beta;\mu\nu;\omega_n)\mathcal{G}(\mu\nu;\gamma\delta;\omega_n) \tag{129}$$

where $K(\alpha\beta;\mu\nu;\omega_n)$ is called interaction kernel. Thus, Eq. (128) can be written in a closed form

$$(i\omega_n-\theta_\alpha\varepsilon_\alpha+\theta_\beta\varepsilon_\beta)\mathcal{G}(\alpha\beta;\gamma\delta;\omega_n)=-S(\alpha\beta;\gamma\delta)+\sum_{\mu\nu}K(\alpha\beta;\mu\nu;\omega_n)\mathcal{G}(\mu\nu;\gamma\delta;\omega_n). \tag{130}$$

Now, let us turn to the equation satisfied by $q\overline{q}$ bound states. This equation can be derived from Eq. (130) with the aid of the following Lehmann representation of the four-point Green function which may be derived by expanding the time-ordered product in Eq. (95) and then inserting the complete set of $q\overline{q}$ bound states into Eq. (95) ,

$$\mathcal{G}(\alpha\beta;\gamma\delta;\omega_l)=\frac{1}{2}e^{\beta\Omega}\sum_{mn}\Delta_{mn}\{\frac{\chi_{nm}(\alpha\beta)\chi_{mn}(\gamma\delta)}{i\omega_l-E_{nm}}-\frac{\chi_{nm}(\gamma\delta)\chi_{mn}(\alpha\beta)}{i\omega_l+E_{nm}}\} \tag{131}$$

where

$$\chi_{nm}(\alpha\beta)=\left\langle m\left|N[\widehat{b}_\alpha\widehat{b}_\beta^+]\right|n\right\rangle \tag{132}$$

which is the transition amplitude from the state with energy E_n to the state with energy E_m and

$$\Delta_{nm}=e^{-\beta E_n}-e^{-\beta E_m}. \tag{133}$$

Upon substituting Eq. (131) into Eq. (130) and then taking the limit: $\lim_{i\omega_l\to E_{nm}}(i\omega_l-E_{nm})$, we get the following equation satisfied by the transition amplitude

$$(E_{nm}-\theta_\alpha\varepsilon_\alpha+\theta_\beta\varepsilon_\beta)\chi_{nm}(\alpha\beta)=\sum_{\gamma\delta}K(\alpha\beta;\gamma\delta;E_{nm})\chi_{nm}(\gamma\delta) \tag{134}$$

where the fact that the function $S(\alpha\beta;\gamma\delta)$ has no bound state poles has been considered. If we take $\mid m\rangle$ to be the vacuum state $\mid 0\rangle$ and set $E=E_{n0}$ and $\chi_n(\alpha\beta)=\left\langle 0\left|N[\widehat{b}_\alpha\widehat{b}_\beta^+]\right|n\right\rangle$, we can write from the above equation that

$$(E-\theta_\alpha\varepsilon_\alpha+\theta_\beta\varepsilon_\beta)\chi_n(\alpha\beta)=\sum_{\gamma\delta}K(\alpha\beta;\gamma\delta;E)\chi_n(\gamma\delta). \tag{135}$$

where the subscript n in E_n has been suppressed. This just is the equation satisfied by the $q\bar{q}$ bound states at finite temperature.

Since the index α contains $\theta_\alpha = \pm$, Eq. (135) actually is a set of coupled equations for the amplitudes $\chi_n(\alpha^+\beta^-)$, $\chi_n(\alpha^-\beta^+)$, $\chi_n(\alpha^+\beta^+)$ and $\chi_n(\alpha^-\beta^-)$. Following the procedure described in Refs. (16) and (17), one may reduce the above equation to an equivalent equation satisfied by the amplitude of positive energy. We do not repeat the derivation here. We only show the result as follows:

$$[E - \varepsilon(\vec{k}_\alpha) - \varepsilon(\vec{k}_\beta)]\psi(\alpha\beta;E) = \sum_{\gamma\delta} V(\alpha\beta;\gamma\delta;E)\psi(\gamma\delta;E). \tag{136}$$

where $\psi(\alpha\beta;E) = \chi_n(\alpha^+\beta^-)$ and $V(\alpha\beta;\gamma\delta;E)$ is the interaction Hamiltonian which can be expressed as

$$V(\alpha\beta;\gamma\delta;E) = \sum_{n=0} V^{(n)}(\alpha\beta;\gamma\delta;E), \tag{137}$$

in which

$$V^{(0)}(\alpha\beta;\gamma\delta;E) = K_{++++}(\alpha\beta;\gamma\delta;E), \tag{138}$$

$$V^{(1)}(\alpha\beta;\gamma\delta;E) = \sum_{ab\neq ++} \sum_{\rho\sigma} \frac{K_{++ab}(\alpha\beta;\rho\sigma;E)K_{ab++}(\rho\sigma;\gamma\delta;E)}{E - a\varepsilon(\vec{k}_\rho) - b\varepsilon(\vec{k}_\sigma)}, \tag{139}$$

$$\begin{aligned} &V^{(2)}(\alpha\beta;\gamma\delta;E) \\ &= \sum_{ab\neq ++} \sum_{cd\neq ++} \sum_{\rho\sigma} \sum_{\mu\nu} \frac{K_{++ab}(\alpha\beta;\rho\sigma;E)K_{abcd}(\rho\sigma;\mu\nu;E)K_{cd++}(\mu\nu;\gamma\delta;E)}{(E-a\varepsilon(\vec{k}_\rho)-b\varepsilon(\vec{k}_\sigma))(E-c\varepsilon(\vec{k}_\mu)-d\varepsilon(\vec{k}_\nu))}, \\ &\cdots\cdots. \end{aligned} \tag{140}$$

here $a, b = \pm$, and

$$\begin{aligned} K_{++++}(\alpha\beta;\gamma\delta;E) &= K(\alpha^+\beta^-;\gamma^+\delta^-;E), \\ K_{----}(\alpha\beta;\gamma\delta;E) &= K(\alpha^-\beta^+;\gamma^-\delta^+;E) \\ K_{+-+-}(\alpha\beta;\gamma\delta;E) &= K(\alpha^+\beta^+;\gamma^+\delta^+;E), \\ K_{-+-+}(\alpha\beta;\gamma\delta;E) &= K(\alpha^-\beta^-;\gamma^-\delta^-;E). \end{aligned} \tag{141}$$

IV. CLOSED EXPRESSION OF THE INTERACTION KERNEL IN THE EQUATION FOR $Q\overline{Q}$ BOUND STATES

In this section, we are devoted to deriving a closed expression of the interaction kernel appearing in Eq. (135) and defined in Eq. (127). For this derivation, we need equations of motion which describe evolution of the Green functions $\mathcal{G}(\alpha\beta;\gamma\delta;\tau_1-\tau_2)$ and $\mathcal{G}(\alpha\beta\sigma;\gamma\delta;\tau_1-\tau_2)$ with time τ_2. Taking the derivatives of the generating functional in Eq. (100) with respect to $b_\gamma^*(\tau_2)$ and $b_\delta(\tau_2)$ respectively, by the same procedure as described in the derivation of Eq. (103), one may obtain

$$\{\frac{d}{d\tau_2}\frac{\delta}{\delta\eta_\gamma^*(\tau_2)} + \theta_\gamma\varepsilon_\gamma\frac{\delta}{\delta\eta_\gamma^*(\tau_2)} + \sum_{\rho\lambda}\theta_\gamma\theta_\rho A(\gamma\rho\lambda)\frac{\delta^2}{\delta\eta_\rho^*(\tau_2)\delta j_\lambda^*(\tau_2)} - \eta_\gamma(\tau_2)\}Z[j] = 0. \tag{142}$$

and

$$\{\frac{d}{d\tau_2}\frac{\delta}{\delta\eta_\delta(\tau_2)} - \theta_\delta\varepsilon_\delta\frac{\delta}{\delta\eta_\delta(\tau_2)} - \sum_{\sigma\lambda}\theta_\delta\theta_\sigma A(\sigma\delta\lambda)\frac{\delta^2}{\delta\eta_\sigma(\tau_2)\delta j_\lambda^*(\tau_2)} - \eta_\delta^*(\tau_2)\}Z[j] = 0. \tag{143}$$

Performing differentiations of Eqs. (142) and (143) with respect to the sources $\eta_\delta(\tau_2)$ and $\eta_\gamma^*(\tau_2)$ respectively, we get

$$\begin{aligned} \{(\tfrac{d}{d\tau_2}\tfrac{\delta}{\delta\eta_\gamma^*(\tau_2)})\tfrac{\delta}{\delta\eta_\delta(\tau_2)} + \theta_\gamma\varepsilon_\gamma\tfrac{\delta^2}{\delta\eta_\gamma^*(\tau_2)\delta\eta_\delta(\tau_2)} + \sum_{\rho\lambda}\theta_\gamma\theta_\rho A(\gamma\rho\lambda)\tfrac{\delta^3}{\delta\eta_\rho^*(\tau_2)\delta\eta_\delta(\tau_2)\delta j_\lambda^*(\tau_2)} \\ +\delta_{\gamma\delta} - \eta_\gamma(\tau_2)\tfrac{\delta}{\delta\eta_\delta(\tau_2)}\}Z[j] = 0 \end{aligned} \tag{144}$$

and

$$\{\frac{\delta}{\delta\eta_\gamma^*(\tau_2)}(\frac{d}{d\tau_2}\frac{\delta}{\delta\eta_\delta(\tau_2)})-\theta_\delta\varepsilon_\delta\frac{\delta}{\delta\eta_\gamma^*(\tau_2)\delta\eta_\delta(\tau_2)}-\sum_{\sigma\lambda}\theta_\delta\theta_\sigma A(\sigma\delta\lambda)\frac{\delta^3}{\delta\eta_\gamma^*(\tau_2)\delta\eta_\sigma(\tau_2)\delta j_\lambda^*(\tau_2)} \\ -\delta_{\gamma\delta}+\eta_\delta^*(\tau_2)\frac{\delta}{\delta\eta_\gamma^*(\tau_2)}\}Z[j]=0. \tag{145}$$

Furthermore, by successively differentiating Eqs. (144) and (145) with respect to the sources $\eta_\alpha^*(\tau_1)$ and $\eta_\beta(\tau_1)$, one obtains

$$\{\frac{\delta^2}{\delta\eta_\alpha^*(\tau_1)\delta\eta_\beta(\tau_1)}(\frac{d}{d\tau_2}\frac{\delta}{\delta\eta_\gamma^*(\tau_2)})\frac{\delta}{\delta\eta_\delta(\tau_2)}+\theta_\gamma\varepsilon_\gamma\frac{\delta^4}{\delta\eta_\alpha^*(\tau_1)\delta\eta_\beta(\tau_1)\delta\eta_\gamma^*(\tau_2)\delta\eta_\delta(\tau_2)} \\ +\sum_{\lambda\sigma}\theta_\gamma\theta_\lambda A(\gamma\rho\lambda)\frac{\delta^5}{\delta\eta_\alpha^*(\tau_1)\delta\eta_\beta(\tau_1)\delta\eta_\rho^*(\tau_2)\delta\eta_\delta(\tau_2)\delta j_\lambda^*(\tau_2)}+\delta_{\gamma\delta}\frac{\delta^2}{\delta\eta_\alpha^*(\tau_2)\delta\eta_\beta(\tau_2)} \\ -\delta_{\beta\gamma}\delta(\tau_1-\tau_2)\frac{\delta^2}{\delta\eta_\alpha^*(\tau_1)\delta\eta_\delta(\tau_2)}-\eta_\gamma(\tau_2)\frac{\delta^3}{\delta\eta_\alpha^*(\tau_1)\delta\eta_\beta(\tau_1)\delta\eta_\delta(\tau_2)}\}Z[j]=0 \tag{146}$$

and

$$\{\frac{\delta^3}{\delta\eta_\alpha^*(\tau_1)\delta\eta_\beta(\tau_1)\delta\eta_\gamma^*(\tau_2)}(\frac{d}{d\tau_2}\frac{\delta}{\delta\eta_\delta(\tau_2)})-\theta_\delta\varepsilon_\delta\frac{\delta^4}{\delta\eta_\alpha^*(\tau_1)\delta\eta_\beta(\tau_1)\delta\eta_\gamma^*(\tau_2)\delta\eta_\delta(\tau_2)} \\ -\sum_{\sigma\lambda}\theta_\delta\theta_\sigma A(\sigma\delta\lambda)\frac{\delta^5}{\delta\eta_\alpha^*(\tau_1)\delta\eta_\beta(\tau_1)\delta\eta_\gamma^*(\tau_2)\delta\eta_\sigma(\tau_2)\delta j_\lambda^*(\tau_2)}-\delta_{\gamma\delta}\frac{\delta^2}{\delta\eta_\alpha^*(\tau_2)\delta\eta_\beta(\tau_2)} \\ +\delta_{\alpha\delta}\delta(\tau_1-\tau_2)\frac{\delta^2}{\delta\eta_\gamma^*(\tau_2)\delta\eta_\beta(\tau_1)}+\eta_\delta^*(\tau_2)\frac{\delta^3}{\delta\eta_\alpha^*(\tau_1)\delta\eta_\beta(\tau_1)\delta\eta_\gamma^*(\tau_2)}\}Z[j]=0. \tag{147}$$

Let us sum up Eqs. (144) and (145) at first, then multiply the both sides of the equation thus obtained with $-\theta_\gamma\theta_\delta$ and finally set all the sources but the source j_λ to vanish. By these operations, we get

$$\sum_{\rho\sigma}\overline{H}(\gamma\delta;\rho\sigma;\tau_2)^{j_\lambda}S_{\rho\sigma}^{j_\lambda}=0 \tag{148}$$

where

$$\overline{H}(\gamma\delta;\rho\sigma;\tau_2)^{j_\lambda}=(\frac{d}{d\tau_2}+\theta_\gamma\varepsilon_\gamma-\theta_\delta\varepsilon_\delta)\delta_{\gamma\rho}\delta_{\delta\sigma}-\sum_\lambda f(\rho\sigma\lambda;\gamma\delta)\frac{\delta}{\delta j_\lambda^*(\tau_2)} \tag{149}$$

in which

$$f(\rho\sigma\lambda;\gamma\delta)=A(\sigma\delta\lambda)\delta_{\gamma\rho}-A(\gamma\rho\lambda))\delta_{\delta\sigma}=-f(\gamma\delta;\rho\sigma\lambda) \tag{150}$$

and $S_{\rho\sigma}^{j_\lambda}$ was defined in Eq. (110).

When we sum up Eqs. (146) and (147), then multiply the both sides of the equation thus obtained with $\theta_\alpha\theta_\beta\theta_\gamma\theta_\delta$ and finally set all the sources but the source j_λ to be zero, according to the definitions in Eqs. (115) and (116), it is found that

$$\sum_{\rho\sigma}\overline{H}(\gamma\delta;\rho\sigma;\tau_2)^{j_\lambda}G(\alpha\beta;\gamma\delta;\tau_1-\tau_2)^{j_\lambda}=\delta(\tau_1-\tau_2)[\delta_{\alpha\delta}S_{\gamma\beta}(\tau_2-\tau_1)^{j_\lambda}-\delta_{\beta\gamma}S_{\alpha\delta}(\tau_1-\tau_2)^{j_\lambda}]. \tag{151}$$

In order to derive the equation of motion satisfied by the Green function $G(\lambda\tau\sigma;\gamma\delta;\tau_1-\tau_2)$ defined in Eq. (126), we may take the derivative of Eq. (151) with respect to $j_\lambda^*(\tau_1)$. In this way, we get

$$\sum_{\rho\sigma}\overline{H}(\gamma\delta;\rho\sigma;\tau_2)^{j_\lambda}G(\alpha\beta\lambda;\gamma\delta;\tau_1-\tau_2)^{j_\lambda}=\delta(\tau_1-\tau_2)[\delta_{\alpha\delta}\Lambda(\gamma\beta\rho;\tau_2-\tau_1)^{j_\lambda}-\delta_{\beta\gamma}\Lambda(\alpha\delta\rho;\tau_1-\tau_2)^{j_\lambda}] \tag{152}$$

where

$$\Lambda(\gamma\beta\rho;\tau_2-\tau_1)^{j_\lambda}=\frac{\delta}{\delta j_\lambda^*(\tau_1)}S_{\gamma\beta}(\tau_2-\tau_1)^{j_\lambda}, \tag{153}$$

$$\Lambda(\alpha\delta\rho;\tau_1-\tau_2)^{j_\lambda}=\frac{\delta}{\delta j_\lambda^*(\tau_1)}S_{\alpha\delta}(\tau_1-\tau_2)^{j_\lambda} \tag{154}$$

and

$$G(\alpha\beta\lambda;\gamma\delta;\tau_1-\tau_2)=\frac{\delta}{\delta j^*_\lambda(\tau_1)}G(\alpha\beta;\gamma\delta;\tau_1-\tau_2)^{j_\lambda}. \tag{155}$$

Acting on Eqs. (98) and (125) with the operator $\overline{H}(\gamma\delta;\rho\sigma;\tau_2)$ and employing Eq. (148), we find

$$\sum_{\rho\sigma}\overline{H}(\gamma\delta;\rho\sigma;\tau_2)^{j_\lambda}\mathcal{G}(\alpha\beta;\rho\sigma;\tau_1-\tau_2)^{j_\lambda}=\sum_{\rho\sigma}\overline{H}(\gamma\delta;\rho\sigma;\tau_2)^{j_\lambda}G(\alpha\beta;\rho\sigma;\tau_1-\tau_2)^{j_\lambda} \tag{156}$$

and

$$\sum_{\rho\sigma}\overline{H}(\gamma\delta;\rho\sigma;\tau_2)^{j_\lambda}\mathcal{G}(\alpha\beta\lambda;\rho\sigma;\tau_1-\tau_2)^{j_\lambda}=\sum_{\rho\sigma}\overline{H}(\gamma\delta;\rho\sigma;\tau_2)^{j_\lambda}G(\alpha\beta\lambda;\rho\sigma;\tau_1-\tau_2)^{j_\lambda}. \tag{157}$$

The above two equalities further indicate that the equations of motion satisfied by the Green functions $\mathcal{G}(\alpha\beta;\rho\sigma;\tau_1-\tau_2)^{j_\lambda}$ and $\mathcal{G}(\alpha\beta\lambda;\rho\sigma;\tau_1-\tau_2)^{j_\lambda}$ are formally the same as those for the ordinary Green functions $G(\alpha\beta;\rho\sigma;\tau_1-\tau_2)^{j_\lambda}$ and $G(\alpha\beta\lambda;\rho\sigma;\tau_1-\tau_2)^{j_\lambda}$ respectively. Upon inserting Eqs. (151) into Eq. (156) and Eq. (152) into Eq. (157) and turning off the source j_λ, noticing the definition in Eq. (149), we derive the following equations

$$\begin{aligned}(\tfrac{d}{d\tau_2}+\theta_\gamma\varepsilon_\gamma-\theta_\delta\varepsilon_\delta)\mathcal{G}(\alpha\beta;\gamma\delta;\tau_1-\tau_2)=\delta(\tau_1-\tau_2)[\delta_{\alpha\delta}S_{\gamma\beta}(\tau_2-\tau_1)\\ -\delta_{\beta\gamma}S_{\alpha\delta}(\tau_1-\tau_2)]+\sum_{\lambda\tau\sigma}\mathcal{G}(\alpha\beta;\lambda\tau\sigma;\tau_1-\tau_2)f(\lambda\tau\sigma;\gamma\delta)\end{aligned} \tag{158}$$

and

$$\begin{aligned}(\tfrac{d}{d\tau_2}+\theta_\gamma\varepsilon_\gamma-\theta_\delta\varepsilon_\delta)\mathcal{G}(\alpha\beta\rho;\gamma\delta;\tau_1-\tau_2)=\delta(\tau_1-\tau_2)[\delta_{\alpha\delta}\Lambda(\gamma\beta\rho;\tau_2-\tau_1)\\ -\delta_{\beta\gamma}\Lambda(\alpha\delta\rho;\tau_1-\tau_2)]+\sum_{\lambda\tau\sigma}\mathcal{G}(\alpha\beta\rho;\lambda\tau\sigma;\tau_1-\tau_2)f(\lambda\tau\sigma;\gamma\delta)\end{aligned} \tag{159}$$

where some indices have been changed for convenience,

$$\begin{aligned}\Lambda(\gamma\beta\rho;\tau_2-\tau_1)=\left\langle T[\widehat{b}_\gamma(\tau_2)\widehat{b}^+_\beta(\tau_1)\widehat{a}_\rho(\tau_1)]\right\rangle_\beta\\ \Lambda(\alpha\delta\rho;\tau_1-\tau_2)=\left\langle T[\widehat{b}_\alpha(\tau_1)\widehat{b}^+_\delta(\tau_2)\widehat{a}_\rho(\tau_1)]\right\rangle_\beta\end{aligned} \tag{160}$$

which are given by Eqs. (153) and (154) with setting $j_\lambda=0$ and

$$\begin{aligned}\mathcal{G}(\lambda\tau\rho;\gamma\delta\sigma;\tau_1-\tau_2)=\tfrac{\delta^2}{\delta j^*_\rho(\tau_1)\delta j^*_\sigma(\tau_2)}\mathcal{G}(\lambda\tau;\gamma\delta;\tau_1-\tau_2)^{j_\lambda}\mid_{j_\lambda=0}\\ =\left\langle T\{N[\widehat{b}_\lambda(\tau_1)\widehat{b}^+_\tau(\tau_1)\widehat{a}_\rho(\tau_1)]N[\widehat{b}_\gamma(\tau_2)\widehat{b}^+_\delta(\tau_2)\widehat{a}_\sigma(\tau_2)]\}\right\rangle_\beta\end{aligned} \tag{161}$$

is the six-point Green function including two gluon operators in it. According to the definition in Eq. (123), we have

$$\mathcal{G}(\lambda\tau\rho;\gamma\delta\sigma;\tau_1-\tau_2)=G(\lambda\tau\rho;\gamma\delta\sigma;\tau_1-\tau_2)-\Lambda(\lambda\tau\rho)\Lambda(\gamma\delta\sigma) \tag{162}$$

where

$$G(\lambda\tau\rho;\gamma\delta\sigma;\tau_1-\tau_2)=\langle T[\widehat{b}_\lambda(\tau_1)\widehat{b}^+_\tau(\tau_1)\widehat{a}_\rho(\tau_1)\widehat{b}_\gamma(\tau_2)\widehat{b}^+_\delta(\tau_2)\widehat{a}_\sigma(\tau_2)]\rangle_\beta \tag{163}$$

is the ordinary six-point Green function. It should be noted that due to the restriction of the delta function, the terms in the brackets on the right hand sides of Eqs. (158) and (159) actually are "time"-independent.

It is easy to see that the Green functions $\mathcal{G}(\alpha\beta;\lambda\tau\sigma;\tau_1-\tau_2)$ and $\mathcal{G}(\lambda\tau\rho;\gamma\delta\sigma;\tau_1-\tau_2)$, as the Green functions $\mathcal{G}(\alpha\beta;\gamma\delta;\tau_1-\tau_2)$ and $\mathcal{G}(\alpha\beta\rho;\gamma\delta;\tau_1-\tau_2)$, are periodic. Therefore, by the Fourier transformation, i.e. by the integration $\int_0^\beta d\tau e^{i\omega_n\tau}$, noticing $d/d\tau_2=-d/d\tau$, Eqs. (158) and (159) will be transformed to

$$\begin{aligned}(i\omega_n+\theta_\gamma\varepsilon_\gamma-\theta_\delta\varepsilon_\delta)\mathcal{G}(\alpha\beta;\gamma\delta;\omega_n)=S(\alpha\beta;\gamma\delta)\\ +\sum_{\lambda\tau\sigma}\mathcal{G}(\alpha\beta;\lambda\tau\sigma;\omega_n)f(\lambda\tau\sigma;\gamma\delta)\end{aligned} \tag{164}$$

where $S(\alpha\beta;\gamma\delta)$ was defined in Eq. (122) and

$$\begin{aligned}(i\omega_n+\theta_\gamma\varepsilon_\gamma-\theta_\delta\varepsilon_\delta)\mathcal{G}(\alpha\beta\rho;\gamma\delta;\omega_n)=R(\alpha\beta\rho;\gamma\delta)\\ +\sum_{\lambda\tau\sigma}\mathcal{G}(\alpha\beta\rho;\lambda\tau\sigma;\omega_n)f(\lambda\tau\sigma;\gamma\delta)\end{aligned} \tag{165}$$

where

$$\begin{aligned} R(\alpha\beta\rho;\gamma\delta) &= \delta_{\alpha\delta}\Lambda(\gamma\beta\rho) - \delta_{\beta\gamma}\Lambda(\alpha\delta\rho) \\ &= \langle[\widehat{b}_\alpha\widehat{b}^+_\beta, \widehat{b}_\gamma\widehat{b}^+_\delta]_-\widehat{a}_\rho\rangle_\beta \end{aligned} \tag{166}$$

which is "time"-independent.

Now we are ready to derive the interaction kernel. Acting on the both sides of Eq. (129) with $(i\omega_n + \theta_\gamma\varepsilon_\gamma - \theta_\delta\varepsilon_\delta)$ and using Eqs. (164) and (165), one gets

$$\begin{aligned} \sum_{\mu\nu} K(\alpha\beta;\mu\nu;\omega_n)S(\mu\nu;\gamma\delta) &= \sum_{\lambda\tau\rho} f(\alpha\beta;\lambda\tau\rho)R(\lambda\tau\rho;\gamma\delta) + \sum_{\lambda\tau\rho}\sum_{\xi\eta\sigma} f(\alpha\beta;\lambda\tau\rho) \\ \times\mathcal{G}(\lambda\tau\rho;\xi\eta\sigma;\omega_n)f(\xi\eta\sigma;\gamma\delta) &- \sum_{\mu\nu}\sum_{\xi\eta\sigma} K(\alpha\beta;\mu\nu;\omega_n)\mathcal{G}(\alpha\beta;\xi\eta\sigma;\omega_n)f(\lambda\tau\sigma;\gamma\delta). \end{aligned} \tag{167}$$

Operating on the both sides of Eq. (129) with the inverse of $\mathcal{G}(\mu\nu;\gamma\delta;\omega_n)$, we have

$$K(\alpha\beta;\gamma\delta;\omega_n) = \sum_{\gamma\delta}\sum_{\lambda\tau\sigma} f(\alpha\beta;\lambda\tau\rho)\mathcal{G}(\lambda\tau\rho;\mu\nu;\omega_n)\mathcal{G}^{-1}(\mu\nu;\gamma\delta;\omega_n). \tag{168}$$

Upon substituting Eq. (168) onto the right hand side of Eq. (167) and acting on Eq. (167) with the inverse $S^{-1}(\mu\nu;\gamma\delta)$, we eventually arrive at

$$\begin{aligned} K(\alpha\beta;\gamma\delta;E) &= \sum_{\mu\nu}\{\sum_{\lambda\tau\rho} f(\alpha\beta;\lambda\tau\rho)R(\lambda\tau\rho;\mu\nu) + \sum_{\lambda\tau\rho}\sum_{\xi\eta\sigma} f(\alpha\beta;\lambda\tau\rho)\mathcal{G}(\lambda\tau\rho;\xi\eta\sigma;E)f(\xi\eta\sigma;\mu\nu) \\ &- \sum_{\lambda\tau\rho}\sum_{\xi\eta\sigma}\sum_{\kappa\varsigma}\sum_{\pi\theta} f(\alpha\beta;\lambda\tau\rho)\mathcal{G}(\lambda\tau\rho;\kappa\varsigma;E)\mathcal{G}^{-1}(\kappa\varsigma;\pi\theta;E)\mathcal{G}(\pi\theta;\xi\eta\sigma;E)f(\xi\eta\sigma;\mu\nu)\}S^{-1}(\mu\nu;\gamma\delta) \end{aligned} \tag{169}$$

where ω_n has been replaced by E. This just is the wanted closed expression of the interaction kernel appearing in Eq. (135). In accordance with Eq. (129), the last term in Eq. (169) can be written in the form

$$\sum_{\rho\sigma}\sum_{\xi\eta}\sum_{\mu\nu} K(\alpha\beta;\rho\sigma;E)\mathcal{G}(\rho\sigma;\xi\eta;E)K(\xi\eta;\mu\nu;E)S^{-1}(\mu\nu;\gamma\delta) \tag{170}$$

which exhibits a typical B-S reducible structure. Therefore, the last term in Eq. (16) plays the role of cancelling the B-S reducible part contained in the other terms in Eq. (169) to make the kernel to be B-S irreducible. If we use the above expression in place of the last term in Eq. (169) and acting on Eq. (169) with $S(\gamma\delta;\mu\nu)$, we obtain from Eq. (169) an integral equation satisfied by the kernel $K(\alpha\beta;\gamma\delta;E)$. Define

$$\mathcal{R}(\alpha\beta;\gamma\delta) = \sum_{\lambda\tau\rho} f(\alpha\beta;\lambda\tau\rho)R(\lambda\tau\rho;\gamma\delta) \tag{171}$$

and

$$\mathcal{Q}(\alpha\beta;\gamma\delta) = \sum_{\lambda\tau\rho}\sum_{\xi\eta\sigma} f(\alpha\beta;\lambda\tau\rho)\mathcal{G}(\lambda\tau\rho;\xi\eta\sigma;E)f(\xi\eta\sigma;\gamma\delta), \tag{172}$$

the integral equation can be written in the matrix form as follows

$$KS = \mathcal{R} + \mathcal{Q} - K\mathcal{G}K. \tag{173}$$

For comparison with the kernel in Eq. (169) and for convenience of nonperturbative investigations, we would like to show the corresponding closed expression given in the position space without giving derivation. This kernel can be obtained from the kernel in Eq. (169) by making use of the inverse of the Fourier transformations written in sect. 3 or derived from the generating functional represented in the position space by completely following the procedure as described in this section. The kernel is represented as follows:

$$\begin{aligned} &K(\vec{x}_1,\vec{x}_2;\vec{y}_1,\vec{y}_2;E) = \int d^3z_1 d^3z_2\{\mathcal{R}(\vec{x}_1,\vec{x}_2;\vec{z}_1,\vec{z}_2) \\ &+\mathcal{Q}(\vec{x}_1,\vec{x}_2;\vec{z}_1,\vec{z}_2;E) - \mathcal{D}(\vec{x}_1,\vec{x}_2;\vec{z}_1,\vec{z}_2;E)\}S^{-1}(\vec{z}_1,\vec{z}_2;\vec{y}_1,\vec{y}_2) \end{aligned} \tag{174}$$

where $\mathcal{R}(\vec{x}_1,\vec{x}_2;\vec{z}_1,\vec{z}_2)$, $\mathcal{Q}(\vec{x}_1,\vec{x}_2;\vec{z},,\vec{z}_2;E)$ and $\mathcal{D}(\vec{x}_1,\vec{x}_2;\vec{z}_1,\vec{z}_2;E)$ are separately described below.

The function $\mathcal{R}(\vec{x}_1,\vec{x}_2;\vec{z}_1,\vec{z}_2)$ can be represented as

$$\mathcal{R}(\vec{x}_1,\vec{x}_2;\vec{z}_1,\vec{z}_2)=\sum_{i=1}^{2}\Omega_i^{a\mu}\mathcal{R}_\mu^{(i)a}(\vec{x}_1,\vec{x}_2;\vec{z}_1,\vec{z}_2) \tag{175}$$

in which

$$\Omega_1^{a\mu}=ig\gamma_1^4\gamma_1^\mu T_1^a,\ \Omega_2^{b\nu}=ig\gamma_2^4\gamma_2^\nu\overline{T}_2^b \tag{176}$$

with $T_1^a=\lambda^a/2$ and $\overline{T}_2^b=-\lambda^{a*}/2$ being the quark and antiquark color matrices respectively and

$$\mathcal{R}_\mu^{(i)a}(\vec{x}_1,\vec{x}_2;\vec{z}_1,\vec{z}_2)=\delta^3(\vec{x}_1-\vec{z}_1)\gamma_1^4\Lambda_\mu^{\mathbf{c}a}(\vec{x}_i\mid\vec{x}_2,\vec{z}_2)+\delta^3(\vec{x}_2-\vec{z}_2)\gamma_2^4\Lambda_\mu^a(\vec{x}_i\mid\vec{x}_1,\vec{z}_1) \tag{177}$$

here $\Lambda_\mu^a(\vec{x}_i\mid\vec{x}_1,\vec{y}_1)$ and $\Lambda_\mu^{\mathbf{c}a}(\vec{x}_i\mid\vec{x}_2,\vec{y}_2)$ are defined as

$$\begin{aligned}\Lambda_\mu^a(\vec{x}_i\mid\vec{x}_1,\vec{y}_1)&=\langle T[\mathbf{A}_\mu^a(\vec{x}_i,\tau_1)\psi(\vec{x}_1,\tau_1)\overline{\psi}(\vec{y}_1,\tau_1)]\rangle_\beta,\\ \Lambda_\mu^{\mathbf{c}a}(\vec{x}_i\mid\vec{x}_2,\vec{y}_2)&=\langle T[\mathbf{A}_\mu^a(\vec{x}_i,\tau_1)\psi^c(\vec{x}_2,\tau_1)\overline{\psi}^c(\vec{y}_2,\tau_1)]\rangle_\beta\end{aligned} \tag{178}$$

which are time-independent due to the translation-invariance property of the Green functions.

The function is of the form

$$\mathcal{Q}(\vec{x}_1,\vec{x}_2;\vec{z}_1,\vec{z}_2;E)=\sum_{i,j=1}^{2}\Omega_i^{a\mu}\mathcal{G}_{\mu\nu}^{ab}(\vec{x}_i,\vec{z}_j\mid\vec{x}_1,\vec{x}_2;\vec{z}_1,\vec{z}_2;E)\overline{\Omega}_j^{b\nu} \tag{179}$$

in which

$$\overline{\Omega}_1^{a\mu}=ig\gamma_1^\mu\gamma_1^4T_1^a,\ \overline{\Omega}_2^{a\mu}=ig\gamma_2^\mu\gamma_2^4\overline{T}_2^a, \tag{180}$$

$\mathcal{G}_{\mu\nu}^{ab}(\vec{x}_i,\vec{z}_j\mid\vec{x}_1,\vec{x}_2;\vec{z}_1,\vec{z}_2;E)$ is the Fourier transform of the Green function defined by

$$\begin{aligned}&\mathcal{G}_{\mu\nu}^{ab}(\vec{x}_i,\vec{z}_j\mid\vec{x}_1,\vec{x}_2;\vec{z}_1,\vec{z}_2;\tau_1-\tau_2)\\&=\langle T\{N[\mathbf{A}_\mu^a(\vec{x}_i,\tau_1)\psi(\vec{x}_1,\tau_1)\psi^c(\vec{x}_2,\tau_1)]N[\mathbf{A}_\nu^b(\vec{z}_j,\tau_2)\overline{\psi}(\vec{z}_1,\tau_2)\overline{\psi}^c(\vec{z}_2,\tau_2)]\}\rangle_\beta\end{aligned} \tag{181}$$

The function $\mathcal{D}(\vec{x}_1,\vec{x}_2;\vec{z}_1,\vec{z}_2;E)$ is expressed by

$$\begin{aligned}&\mathcal{D}(\vec{x}_1,\vec{x}_2;\vec{z}_1,\vec{z}_2;E)=\int\prod_{k=1}^{2}d^3u_kd^3v_k\sum_{i,j=1}^{2}\Omega_i^{a\mu}\mathcal{G}_\mu^{(i)a}(\vec{x}_i\mid\vec{x}_1,\vec{x}_2;\vec{u}_1,\vec{u}_2;E)\\&\times\mathcal{G}^{-1}(\vec{u}_1,\vec{u}_2;\vec{v}_1,\vec{v}_2;E)\mathcal{G}_\nu^{(j)b}(\vec{z}_j\mid\vec{v}_1,\vec{v}_2;\vec{z}_1,\vec{z}_2;E)\overline{\Omega}_j^{b\nu}\end{aligned} \tag{182}$$

in which $\mathcal{G}_\mu^{(i)a}(\vec{x}_i\mid\vec{x}_1,\vec{x}_2;\vec{u}_1,\vec{u}_2;E)$ and $\mathcal{G}_\nu^{(j)b}(\vec{z}_j\mid\vec{v}_1,\vec{v}_2;\vec{z}_1,\vec{z}_2;E)$ are the Fourier transforms of the following Green functions

$$\begin{aligned}&\mathcal{G}_\mu^{(i)a}(\vec{x}_i\mid\vec{x}_1,\vec{x}_2;\vec{u}_1,\vec{u}_2;\tau_1-\tau_2)\\&=\langle T\{N[\mathbf{A}_\mu^a(\vec{x}_i,\tau_1)\psi(\vec{x}_1,\tau_1)\psi^c(\vec{x}_2,\tau_1)]N[\overline{\psi}(\vec{u}_1,\tau_2)\overline{\psi}^c(\vec{u}_2,\tau_2)]\}\rangle_\beta\end{aligned} \tag{183}$$

and

$$\begin{aligned}&\mathcal{G}_\nu^{(j)b}(\vec{z}_j\mid\vec{v}_1,\vec{v}_2;\vec{z}_1,\vec{z}_2;\tau_1-\tau_2)\\&=\langle T\{N[\psi(\vec{v}_1,\tau_1)\psi^c(\vec{v}_2,\tau_1)]N[\mathbf{A}_\nu^b(\vec{z}_j,\tau_2)\overline{\psi}(\vec{z}_1,\tau_2)\overline{\psi}^c(\vec{z}_2,\tau_2)]\}\rangle_\beta\end{aligned} \tag{184}$$

The $S^{-1}(\vec{z}_1,\vec{z}_2;\vec{y}_1,\vec{y}_2)$ in Eq. (174) is the inverse of the function defined by

$$S(\vec{x}_1,\vec{x}_2;\vec{z}_1,\vec{z}_2)=\delta^3(\vec{x}_1-\vec{z}_1)\gamma_1^4S_F^c(\vec{x}_2-\vec{z}_2)+\delta^3(\vec{x}_2-\vec{z}_2)\gamma_2^4S_F(\vec{x}_1-\vec{z}_1) \tag{185}$$

in which $S_F(\vec{x}_1-\vec{z}_1)$ and $S_F^c(\vec{x}_2-\vec{z}_2)$ are the equal-time quark and antiquark thermal propagators respectively. It is clear that there is one-to-one correspondence between the both kernels written in Eqs. (169) and (174). It is noted that the interaction kernel derived in this section is nonperturbative because the Green functions included in the kernel are defined in the Heisenberg picture. Perturbative calculations of the kernel can easily be done by using the familiar perturbative expansions of Green functions. .