An Accelerated Multiplicative Iterative Algorithm in Image Reconstruction

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ABSTRACT: Based on the ML-EM (maximum likelihood expectation maximization) algorithm and AWLS (one kind of multiplicative weighted least square) reconstruction, a new algorithm named RMITC (rapid multiplicative iteration with total-count conservation) is proposed. The new method assumes a higher order correction factor and incorporates a total-count conservation constraint to obtain better images reconstructed while achieving a higher speed of convergence. Computer simulated phantom data and real positron emission tomography (PET) transmission data were used to compare the new method with other reconstruction algorithms, such as ML-EM and AWLS. Results demonstrated that the new method is faster and better quantitatively than both ML-EM and AWLS.

II. MULTIPlicative RECONSTRUCTION ALGORITHMS

The ML-EM reconstruction is given by

$$\hat{\lambda}_{\text{new}}(b) = \hat{\lambda}(b) \sum_t p(t, b) \frac{n(t)}{\hat{n}(t)} \hat{n}(t) = \sum_b p(t, b) \hat{\lambda}(b),$$

where $\hat{\lambda}(b), b = 1, 2, \ldots, B$ denotes the current value of box $b$ (the unknown image is discretized into a grid of $B$ "boxes"), $\hat{\lambda}_{\text{new}}(b)$ represents the updated value, $n(t), t = 1, 2, \ldots, T$ is projection detected in detector bin $t$, and $\hat{n}(t)$ is the reprojection of the current estimate in bin $t$. The probability that box $b$ is detected in bin $t$ is denoted by $p(t, b)$ and assumed to be known for all $t$ and $b$ with the normalization: $\sum_t p(t, b) = 1$ for all $b$. In ML-EM Eq. (1), the ratio $n(t)/\hat{n}(t)$ is just a correction factor that is exerted on the current image in a weighted averaging manner to give the updated image.

AWLS (Anderson et al., 1997) is given by

$$\hat{\lambda}_{\text{new}}(b) = \hat{\lambda}(b) \sum_t p(t, b) \left(\frac{n(t)}{\hat{n}(t)}\right)^2,$$

which is obtained by an EM-like algorithm to minimize the weighted LS estimator

$$\phi(\lambda) = \sum_t \frac{[n(t) - \hat{n}(t)]^2}{\hat{n}(t)}.$$
under the Kuhn–Tucker conditions:

$$\frac{\delta \phi}{\delta \lambda_b} = 0 \quad \text{for } b = 1, 2, 3 \ldots B \quad \text{and} \quad \lambda_b > 0 \quad (4)$$

$$\frac{\delta \phi}{\delta \lambda_b} \leq 0 \quad \text{for } b = 1, 2, 3 \ldots B \quad \text{and} \quad \lambda_b = 0 \quad (5)$$

For the LS objective function, one can rewrite the iteration in a scaled gradient manner (Kaufman, 1993):

$$\hat{\lambda}_{\text{new}}(b) = \hat{\lambda}(b) - \frac{\delta \phi[\hat{\lambda}(b)]}{\delta \lambda_b} \quad (6)$$

The algorithm Eq. (2) has three important properties. First, the iterations are non-negative, provided that the initial estimate \(\hat{\lambda}_0(b) \geq 0\), for all \(b\). Second, the objective function Eq. (3) decreases monotonically with increasing iterative number. Finally, the calculated total-count has a bias:

$$\frac{N^*}{k} < \sum_b \hat{\lambda}(b) \leq kN^* \quad k = \max \left\{ \sum_i \hat{a}(t), 1 \right\} \quad (7)$$

where \(N^*\) is the total-count measured. ML-EM possesses the first and second properties, while the total-count is preserved automatically: \(\sum_b \hat{\lambda}(b) = N^*\). Total-counts bias can lead to poor quantitative images. To solve this problem, Anderson et al. (1997) proposed a penalized term \(\tau J(\lambda)\) into \(\phi(\lambda)\) Eq. (3), where \(J(\lambda) = (\sum_b \hat{\lambda}(b) - N^*)^2\). The penalty parameter \(\tau\) is adjusted so that the penalty term \(\tau J(\lambda)\) exerts an appropriate level of influence on the objective function, thus giving the almost total-count conserved AWLS algorithm:

$$\hat{\lambda}_{\text{new}}(b) = \hat{\lambda}(b) \frac{\sum_i \left( p(t, b) \left( \frac{n(t)}{\hat{a}(t)} \right)^2 + 2\tau N^* \right)}{1 + 2\tau \sum_b \hat{\lambda}(b)} \quad (8)$$

From the theoretical point of view, AWLS algorithm Eq. (2) should converge faster than the ML-EM Eq. (1), because AWLS uses the enhanced correction factor \(\left( n(t)/\hat{a}(t) \right)^2\). As a natural conjecture, a new reconstruction algorithm is proposed:

$$\hat{\lambda}_{\text{new}}(b) = \hat{\lambda}(b) \sum_i \left( p(t, b) \left( \frac{n(t)}{\hat{a}(t)} \right)^3 \right. \quad (9)$$

with the objective function

$$\phi(\lambda) = \sum_i \left[ \frac{(\hat{a}(t) - n(t))^3}{2n(t)\hat{a}(t)} + 2 \frac{(\hat{a}(t) - n(t))^2}{\hat{a}(t)} - \frac{(\hat{a}(t) - n(t))^2}{2n(t)} \right] \quad (10)$$

### Table I. Peak values (real value 70) and errors \(E\) by the above methods.

<table>
<thead>
<tr>
<th>Method</th>
<th>ML-EM</th>
<th>AWLS</th>
<th>RMITC</th>
<th>RMITC</th>
<th>RMITC</th>
</tr>
</thead>
<tbody>
<tr>
<td>(n = 8)</td>
<td>(n = 8)</td>
<td>(n = 8)</td>
<td>(n = 5)</td>
<td>(n = 16)</td>
<td></td>
</tr>
<tr>
<td>(f_{\text{peak}}) (relative unit)</td>
<td>51.0</td>
<td>60.0</td>
<td>68.4</td>
<td>63</td>
<td>74.6</td>
</tr>
<tr>
<td>(E) (relative unit)</td>
<td>69.2</td>
<td>42.6</td>
<td>19.5</td>
<td>44</td>
<td>5.5</td>
</tr>
</tbody>
</table>

![Figure 2](image.png)  
**Figure 1.** Two-dimensional Phantom.

![Figure 2](image.png)  
**Figure 2.** Reconstructed results by (a) ML-EM, with \(n = 8\); (b) AWLS, with \(n = 8\) and penalty \(\tau = 0.1/N^*\); (c) RMITC, with \(n = 8\); (d) RMITC, with \(n = 5\); (e) RMITC, with \(n = 16\).
which is a mixture of several objective functions and can be treated as a multiobjective function (Wang and Lu, 1992). Just like Eq. (3), this function is non-negative and has minimal value when \( \hat{n} = n \). Using Eq. (6), one can get Eq. (9) from this multiobjective function. Compared with Eqs. (1) and (2), the correction factor of the new algorithm is \((n(t)/\hat{n}(t))^2\), so its convergent rate is expected to be faster than both ML-EM Eq. (1) and AWLS Eq. (2). From Eq. (10), we can also see that, through LS procedure, the correction is more emphasized where \( \hat{n}(t) \) has larger deflection from \( n(t) \) by the first term \([\hat{n}(t) - n(t)]^2/2n(t)\hat{n}(t)\) in \( \phi(\lambda) \). Simulated Phantom data and real PET transmission data are used to validate this new algorithm in the next section. We can rewrite the ML-EM, AWLS, and the new algorithm in an integrated formula:

\[
\hat{\lambda}_{new}(b) = \hat{\lambda}(b) \sum_{i} p(t, b) \left( \frac{n(t)}{\hat{n}(t)} \right)^m,
\]

where \( m = 1, 2, 3 \) corresponds to the three multiplicative iterative reconstruction algorithms, Eqs. (1), (2), and (9), respectively.

Analog to Eq. (8), one can get the total-count penalized version of Eq. (9) by adding the total-count penalty term into the objective function Eq. (10), but there is no suitable \( \tau \) to obtain both good image quality and fast convergence.

In order to gain both rapid convergence and better quantitative features by the new method, the following total-count conservation constraint is assumed:

\[
\hat{\lambda}_{new}(b) = \hat{\lambda}(b) \sum_{b'}^{N^*} \lambda(b'),
\]

which is performed in each iteration, in order to give the rescaled updated image. The proposed algorithm together with the total-count conservation constraint is denoted by RMITC (rapid multiplicative iteration with total-count conservation).

To give a quantitative description of the speed of convergence, an error estimator is calculated during each iterative step:

\[
E(n) = \sum_i [n(t) - \hat{n}(t)]^2
\]

III. RESULTS

A 128 \times 128 circular plane phantom [see Eq. (7)] consists of sources (hot spots) with different sizes, as shown in Figure 1. The maximum pixel value is set to 70 (relative unit), with a uniform background of 10. Based on the Poisson model, the projection data are simulated with the expectation equal to the theoretical projection \( \hat{n}(t) \). The total-count is \( 4 \times 10^6 \) and total measured angles is 32 (between \( 0^\circ \sim 180^\circ \)), with 128 bins in the detector array. For reconstruction, all algorithms use the same data and are initialized such that \( \lambda(b) = N^*/B \) for all \( b \).

Figure 2 shows the reconstructed results by the three algorithms (ML-EM, AWLS, and RMITC). By same number of iterative steps \( n = 8 \), the RMITC [Fig. 2(c)] has obviously the highest contrast and best spatial resolution compared to the other two [Fig. 2(a), 2(b)], with the best quantitative feature as well (see Table I).

Comparison of Figure 2(b) and 2(d) shows that the reconstructed image by RMITC with smaller iteration \( n = 5 \) is equivalent to or better than that of AWLS with larger iteration \( n = 8 \) (see Table I, also), whereas the same quality image appears in ML-EM when \( n = 24 \), whose peak value is \( f_{peak} = 67.7 \) and \( E = 16.1 \).

Figure 3 shows the error function \( E \) versus iterative steps of the above reconstruction methods in Figure 2, and fastest convergence of RMITC is seen.

![Figure 3](image3.png)

**Figure 3.** The error function \( E \) versus iterative steps of the above reconstruction methods.

![Figure 4](image4.png)

**Figure 4.** Reconstructed results of real PET thorax transmission data by (a) ML-EM \((n = 5)\) with \( \mu_{peak} = 0.047 \); (b) AWLS \((n = 5)\) with \( \tau = 0.1/N^* \) and \( \mu_{peak} = 0.063 \); (c) RMITC \((n = 5)\) with \( \mu_{peak} = 0.072 \); (d) RMITC \((n = 3)\) with \( \mu_{peak} = 0.060 \); (e) RMITC \((n = 10)\) with \( \mu_{peak} = 0.086 \). (Note: each displayed figure was normalized according to its own maximum.)
As the number of iteration increases, the resulting images by RMITC begin to be speckled, with maximum pixel value overestimated, see Figure 2(e) and Table I, a phenomenon that comes earlier in RMITC than ML-EM and AWLS. So an appropriate stopping rule is needed too in RMITC, as in most iterative methods (Liu and Wang, 1999).

Figure 4 shows the reconstructed results of one section of real PET thorax phantom transmission data. Again, the higher resolution and better quantitative feature are demonstrated by RMITC (note: the linear attenuation coefficient of water at 511 KeV is $\mu = 0.1$).

IV. CONCLUSIONS

To summarize, in the above experimental studies with simulated phantom and real PET data, the reconstructed images we obtained using RMITC have better contrast and resolution than those obtained by ML-EM and AWLS at the same iterative steps. Besides, by adding the total-count conservation constraint, RMITC makes the reconstruction converge to a better quantitative image, while the faster convergence is maintained.

Because the reprojection $\hat{n}(t) = \sum_b p(t, b)\lambda_b$ is the most time-consuming computation, the calculation of correction factors and total-count rescale have not added to the computing burden obviously, so the above algorithms take almost the same computing time per iteration. Some other acceleration mechanisms can also be used in RMITC, such as conjugate gradient, ordered subset techniques, and so on, to speed up the convergence further.

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